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On the In Situ Measurements of the Thermal Conductivity of Deep Sea Sediments

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NOTATION

The Following Notation is used in this Report:

m	=	Water equivalent of 1 cm of probe.
a	=	The radius of the probe.
K	=	Thermal conductivity of the sediment.
k	=	Thermal diffusivity of the sediment.
ρ	=	Density of the sediment.
c	=	The specific heat of the sediment.
t	=	Time.

The theoretical formulae may be expressed in terms of the two dimensionless parameters.

$$\alpha = 2\pi a^2 \rho c/m$$

$$\tau = kt/a^2$$

ABSTRACT

An investigation is made into the accuracy of two asymptotic approximations of the temperature-versus-time curves of a cylindrical probe designed to determine the thermal gradient and thermal conductivity of dee sea sediments. The error resulting from the use of a finite-length probe is discussed.

INTRODUCTION

The analysis of a cylindrical heat source in an infinite medium is used in determining the temperature gradient and the thermal conductivity of ocean bottom sediments. To determine the temperature gradient from a probe that has been forced into the sediments, it is necessary to extrapolate the observed temperature differences to obtain the temperature at equilibrium. The theoretical curve, $F(\alpha, \tau)$ (Bullard, 1954; Jaeger, 1956), decreases rapidly at first with increasing t, approaching zero as $(m/4\pi Kt)$. For small probes this asymptote is an adequate expression of the curve. For larger probes, however, either the earlier portion of the cruve is required or the instrument has to be left in for an excessively long time. The approximation given by Blackwell (1954) can be used for smaller values of t, but this is inadequate for probes of radius greater than 1 cm.

In determining the thermal conductivity, the function $G(\alpha, \tau)$ is used (Jaeger, 1956). For large values of t the temperature, say θ, versus log time (ln t) for a cylindrical heater approaches a linear asymptote with slope equal to $(Q/4\pi K)$, where Q is the heat input. This asymptotic result is sufficient for probes of small diameter such as the "needle probe" (Von Herzen and Maxwell, 1959). In many cases with probes of larger diameter the earlier portion of the curve is required. A number of reduction methods (Blackwell, 1954; Jaeger, 1958) have been proposed, but they all require claculation of theoretical curves for a range of values of the various parameters. Jaeger (1956) calculated a few theoretical curves and more recently Sass (1965) has tabulated $G(\alpha, \tau)$ for almost all useful values of α and τ.

The two functions $F(\alpha, \tau)$ and $G(\alpha, \tau)$ are most conveniently defined in terms of integrals involving Bessel function of the first and second kind and are somewhat tedious to evaluate. Hence it is useful to know when the approximations of Blackwell and the asymptotes[1] can be applied. This report represents tables of: (a) $F(\alpha, \tau)$

[1] We use the word asymptote to designate the first term in an asymptotic expansion.

its Blackwell approximation and asymptotic, $(m/4\pi Kt)$; (b) $G(\alpha, \tau)$ its Blackwell approximation and asymptote $(1/4\pi)\log_e(2.246\tau)$, for various values of α and τ.

EVALUATION OF $G(\alpha, \tau)$ AND $F(\alpha, \tau)$

The theory behind $F(\alpha, \tau)$ and $G(\alpha, \tau)$ is given by Jaeger (1956). $F(\alpha, \tau)$ is defined by:

$$F(\alpha, \tau) = \frac{4\alpha}{\pi^2} \int_0^\infty \frac{\exp(-\tau x^2)\,dx}{x\Delta x} \quad (1)$$

where

$$\Delta(x) = \left[x J_0(x) - \alpha J_1(x)\right]^2 + \left[x Y_0(x) - \alpha Y_1(x)\right]^2 \quad (2)$$

For large values of τ Blackwell (1954)

$$F(\alpha, \tau) \sim \frac{1}{2\alpha\tau} - \frac{1}{4\alpha\tau^2} - \frac{(\alpha - 2)}{4\alpha^2\tau^2}(\log_e 2.246\tau - 1) \quad (3)$$

and the asymptote is:

$$F(\alpha, \tau) \sim \frac{1}{2\alpha\tau} \quad . \quad (4)$$

Equation (1), and subsequent integrals, were evaluated using the Gaussian Legandre quadrature subroutine UCSD GLQUAD and the CDC 3600 computer at the University of California, San Diego. The Bessel functions are most efficiently evaluated by using the economized polynomials presented by Olver (1965).

$G(\alpha, \tau)$ is defined by:

$$G(\alpha, \tau) = \frac{2\alpha^2}{\pi^3} \int_0^\infty \frac{\left\{1 - \exp(-\tau x^2)\right\}}{x^3\,\Delta(x)} \quad (5)$$

For large values of τ,

$$G(\alpha, \tau) \sim \frac{1}{4\pi} \left\{ \log_e 2.246\tau + \frac{1}{2\tau} + \frac{(\alpha - 2)}{2\alpha\tau} \log_e 2.246 \right\} \tag{6}$$

and the asymptote is

$$G(\alpha, \tau) \sim \frac{1}{4\pi} \log_e 2.246\tau$$

For the numerical evaluation, equation (5) was written as:

$$G(\alpha, \tau) = \frac{2\alpha^2}{\pi^3} \left[\int_0^1 \frac{\left[1 - \exp(-\tau x^2)\right] dx}{x^3 \Delta(x)} + \int_0^1 y \frac{\left[1 - \exp(-\tau/y^2)\right] dy}{\Delta(1/y)} \right].$$

TABLES

$F(\alpha, \tau)$ and $G(\alpha, \tau)$, their Blackwell approximations and asymptotes for various values of α and τ are presented in Tables I through V.

The tables indicate that: The Blackwell approximations for $F(\alpha, \tau)$ is within 1% of the exact value when τ is larger than 8; the Blackwell approximation for $G(\alpha, \tau)$ is within 1% of the exact value when τ is larger than 4; when τ is 20, the asymptote to $F(\alpha, \tau)$ is 4% larger than the exact value while the asymptote to $G(\alpha, \tau)$ is within 1% of the exact value. Thus, the Blackwell approximations are significantly better than the asymptotes and can be used with confidence for both $F(\alpha, \tau)$ and $G(\alpha, \tau)$ for τ larger than 8.

THE END EFFECT

During the past two years the transient method of Von Herzen and Maxwell (1959) has bee adapted to measure the <u>in situ</u> thermal conductivity of ocean floor sediments. In order to make the <u>in situ</u> probes sturdy enough to stand up to heave duty at sea, it is desirable that the ratio of diameter-to-length be as large as possible. The theoretical results, however, assume the probe to be of infinite extent. To determine the maximum probe diameter of a given length, which will give readings comparable with the theoretical calculations, it is necessary to know (a) the heat lost by the end, and (b) the time it takes for this effect to travel the length of the probe.

In steady state conditions with constant heat output Q, the effect of the heat loss through the air is given approximately by $a/2\ell$, where $\underline{a}$ is the radius and ℓ is the heater length. For this to be less than one percent, the length should be at least twenty-five times the diameter.

The above figure has been determined assuming a constant rate of heat production and equilibrium conditions. In the actual measurement, the temperature increase of the thermister at the center of the probe is measured only over a short period immediately after the heater current is turned on. Hence, if the distance of the thermistor from the end is long enough, the thermistor will not be effected by the end during the time of measurement. Although the problem has not been investigated in detail, dimensional arguments lead one to conjecture that the time for longitudinal condition along the probe is of the order of $(\ell_1)^2/k_1$, where ℓ_1 is the distance of the thermistor from the end and k_1 is the thermal diffusionary of the probe. For a probe of steel ($k_1 = 0.13\ cm^2/sec$), this time for a 10 cm length is approximately 10 minutes; for a 5 cm length it is only 3 minutes. These figures imply that for a recording time of 10 minues, the thermistor should be at least 10 cm from the end.

For a recording time of 10 minutes and a probe length of 20 cm with the thermistor half way down, the maximum allowable diameter of the probe can be calculated. The "end effect", assuming thermal equilibrium, is $a/2\ell$. After 10 minutes, the thermistor at the center of the probe will be effected an amount $\frac{1}{e} \times \frac{a}{2\ell} \sim \frac{a}{4\ell}$. For this to be less than 1%, $\underline{a}$ must be less than 0.8 cm or the diameter should be less than 1.6 cm. To be on the safe side, a diameter of 1 cm seems perfectly satisfactory. For a thermistor less than 10 cm from the end of the probe, the ℓ/a ratio should be 25 to 1.

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LIST OF TABLES

TABLE I

F (α , τ), G (α , τ), Approx. and Asymptote $\alpha = 1.50$

τ	F (α , τ)	Black. App.	Asymptote
1	.22084	.15606	.33333
2	.13529	.13198	.16667
3	.09756	.09820	.11111
4	.07619	.07707	.08333
5	.06244	.06315	.06667
6	.05285	.05340	.05556
7	.04580	.04621	.04762
8	.04039	.04070	.04167
9	.03611	.03636	.03704
10	.03264	.03284	.03333
11	.02994	.02994	.03030
12	.02751	.02751	.02778
13	.02544	.02544	.02564
14	.02365	.02365	.02381
15	.02210	.02210	.02222
16	.02074	.02074	.02083
17	.01954	.01954	.01961
18	.01847	.01847	.01852
19	.01751	.01751	.01754
20	.01664	.01664	.01667

τ	G (α , τ)	Black. App.	Asymptote
1	.08966	.09343	.06437
2	.13050	.12946	.11953
3	.15781	.15662	.15179
4	.17834	.17735	.17468
5	.19478	.19398	.19243
6	.20848	.20782	.20694
7	.22021	.21967	.21921
8	.23047	.23001	.22983
9	.23959	.23919	.23920
10	.24778	.24744	.24759
11	.25522	.25492	.25517
12	.26203	.26177	.26209
13	.26832	.26808	.26846
14	.27415	.27393	.27436
15	.27959	.27939	.27985
16	.28468	.28450	.28498
17	.28948	.28930	.28981
18	.29400	.29384	.29435
19	.29828	.29813	.29866
20	.30234	.30220	.30274

TABLE II

F (α, τ), G (α, τ), Approx. and Asymptote $\alpha = 2.00$

τ	F (α, τ)	Black. App.	Asymptote
1	.16628	.12500	.25000
2	.09912	.09375	.12500
3	.07088	.06944	.08333
4	.05521	.05469	.06250
5	.04523	.04500	.05000
6	.03830	.03819	.04167
7	.03322	.03316	.03571
8	.02933	.02930	.03125
9	.02625	.02623	.02778
10	.02376	.02375	.02500
11	.02169	.02169	.02273
12	.01997	.01997	.02083
13	.01849	.01849	.01923
14	.01722	.01722	.01786
15	.01611	.01611	.01667
16	.01514	.01514	.01563
17	.01427	.01427	.01471
18	.01350	.01350	.01389
19	.01281	.01281	.01316
20	.01219	.01219	.01250

τ	G (α, τ)	Black. App.	Asymptote
1	.09768	.10416	.06437
2	.13804	.13942	.11953
3	.16459	.16505	.15179
4	.18445	.18462	.17468
5	.20033	.20039	.19243
6	.21357	.21357	.20694
7	.22491	.22489	.21921
8	.23484	.23480	.22983
9	.24367	.24362	.23920
10	.25161	.25156	.24759
11	.25884	.25879	.25517
12	.26546	.26541	.26209
13	.27158	.27152	.26846
14	.27725	.27720	.27436
15	.28255	.28250	.27985
16	.28752	.28747	.28498
17	.29220	.29215	.28981
18	.29662	.29656	.29435
19	.30080	.30075	.29866
20	.30478	.30473	.30274

TABLE III

F (α, τ), G (α, τ), Approx. and Asymptote $\alpha = 2.50$

τ	F (α, τ)	Black. App.	Asymptote
1	.13215	.10382	.20000
2	.07771	.07249	.10000
3	.05540	.05354	.06667
4	.04314	.04226	.05000
5	.03536	.03487	.04000
6	.02997	.02967	.03333
7	.02602	.02581	.02857
8	.02299	.02285	.02500
9	.02060	.02049	.02222
10	.01866	.01858	.02000
11	.01699	.01699	.01818
12	.01565	.01565	.01667
13	.01451	.01451	.01538
14	.01353	.01353	.01429
15	.01267	.01267	.01333
16	.01191	.01191	.01250
17	.01124	.01124	.01176
18	.01064	.01064	.01111
19	.01010	.01010	.01053
20	.00961	.00961	.01000

τ	G (α, τ)	Black. App.	Asymptote
1	.10300	.11060	.06437
2	.14279	.14539	.11953
3	.16876	.17011	.15179
4	.18816	.18899	.17468
5	.20367	.20424	.19243
6	.21661	.21702	.20694
7	.22771	.22802	.21921
8	.23744	.23768	.22983
9	.24609	.24628	.23920
10	.25389	.25404	.24759
11	.26098	.26110	.25517
12	.26749	.26759	.26209
13	.27350	.27359	.26846
14	.27909	.27916	.27436
15	.28431	.28436	.27985
16	.28920	.28925	.28498
17	.29381	.29385	.28981
18	.29817	.29820	.29435
19	.30230	.30232	.29866
20	.30622	.30624	.30274

TABLE IV

F (α , τ), G (α , τ), Approx. and Asymptote $\alpha = 3.00$

τ	F (α , τ)	Black. App.	Asymptote
1	.10913	.08864	.16667
2	.06372	.05901	.08333
3	.04539	.04349	.05556
4	.03535	.03438	.04167
5	.02900	.02842	.03333
6	.02460	.02423	.02778
7	.02137	.02111	.02381
8	.01890	.01871	.02083
9	.01694	.01680	.01852
10	.01535	.01525	.01667
11	.01396	.01396	.01515
12	.01287	.01287	.01389
13	.01194	.01194	.01282
14	.01113	.01113	.01190
15	.01043	.01043	.01111
16	.00981	.00981	.01042
17	.00926	.00926	.00980
18	.00877	.00877	.00926
19	.00833	.00833	.00877
20	.00793	.00793	.00833

τ	F (α , τ)	Black. App.	Asymptote
1	.10677	.11489	.06437
2	.14602	.14938	.11953
3	.17156	.17348	.15179
4	.19063	.19190	.17468
5	.20589	.20680	.19243
6	.21863	.21932	.20694
7	.22957	.23011	.21921
8	.23916	.23959	.22983
9	.24770	.24805	.23920
10	.25539	.25569	.24759
11	.26240	.26265	.25517
12	.26883	.26905	.26209
13	.27478	.27496	.26846
14	.28031	.28046	.27436
15	.28547	.28561	.27985
16	.29032	.29044	.28498
17	.29488	.29499	.28981
18	.29920	.29929	.29435
19	.30329	.30337	.29866
20	.30718	.30725	.30274

TABLE V

$F(\alpha, \tau)$, $G(\alpha, \tau)$, Approx. and Asymptote $\alpha = 4.00$

τ	$F(\alpha, \tau)$	Black. App.	Asymptote
1	.08044	.06847	.12500
2	.04668	.04295	.06250
3	.03326	.03157	.04167
4	.02594	.02501	.03125
5	.02130	.02073	.02500
6	.01809	.01771	.02083
7	.01573	.01546	.01786
8	.01393	.01373	.01563
9	.01250	.01234	.01389
10	.01133	.01122	.01250
11	.01028	.01028	.01136
12	.00948	.00948	.01042
13	.00881	.00881	.00962
14	.00822	.00822	.00893
15	.00771	.00771	.00833
16	.00725	.00725	.00781
17	.00685	.00685	.00735
18	.00649	.00649	.00694
19	.00617	.00617	.00658
20	.00587	.00587	.00625

τ	$F(\alpha, \tau)$	Black. App.	Asymptote
1	.11170	.12025	.06437
2	.15013	.15436	.11953
3	.17508	.17770	.15179
4	.19372	.19554	.17468
5	.20866	.21001	.19243
6	.22114	.22219	.20694
7	.23187	.23272	.21921
8	.24129	.24199	.22983
9	.24968	.25027	.23920
10	.25726	.25775	.24759
11	.26416	.26458	.25517
12	.27050	.27087	.26209
13	.27636	.27668	.26846
14	.28181	.28210	.27436
15	.28691	.28716	.27985
16	.29169	.29192	.28498
17	.29621	.29641	.28981
18	.30047	.30065	.29435
19	.30452	.30468	.29866
20	.30836	.30851	.30274