THE ASTROPHYSICAL JOURNAL, 213:157–164, 1977 April 1

PENETRATION OF THE MOLECULAR-WEIGHT BARRIER

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Received 1975 November 14; revised 1976 September 17

ABSTRACT

We consider the penetration of meridional currents from the radiative envelope of a rotating star into a core with a stable gradient of molecular weight (μ). While the model used is idealized and leaves out a number of effects, it incorporates the main forces and determines the characteristic length and time scales of the motion. In particular, the depth of penetration into a μ-barrier is found.

The results suggest that in spite of a μ-barrier in the core, mixing into the envelope may occur at a nonnegligible rate. The model cannot predict this rate, and instead we propose a recipe to be used in stellar evolution calculations to correct for the mixing.

Subject headings: stars: evolution — stars: interiors — stars: rotation

I. INTRODUCTION

The influence of meridional circulation of the Eddington-Sweet type on the evolution of rotating stars is not generally allowed for in stellar evolution calculations. One reason for this is that nuclear transformations (typically of hydrogen into helium) produce gradients of molecular weight which, as Mestel (1953, 1957) argued, will tend to inhibit the circulation and protect the evolving core from rotational mixing. If the mixing were fast enough, the molecular weight or μ-barrier might not be formed, but this question depends on details of the model, such as shrinking convective cores, and has not yet been fully treated. To discuss these problems, even in particular cases, we need to know when a μ-barrier can actually shield a nuclear burning core from rotational mixing.

That meridional currents can penetrate to some extent into cores with μ-barriers is clear. Indeed, Kippenhahn (1974) has recently discussed an estimate of the circulation speeds in the presence of μ-gradients, and he has also outlined the possible role of such effects in stellar evolution. We compare our results with his in Appendix B.

An analogous problem arises in the theory of solar spin-down. Laminar spin-down currents on a large scale in a stably stratified medium resemble Eddington-Sweet (or, as Kippenhahn calls them, Eddington-Vogt) currents, and have the same time scale. The difference lies in the forcing mechanisms—spin-down currents are driven by solar-wind torques while E-S currents are driven by baroclinity, that is, by horizontal pressure gradients. Here the problem of inhibition of the spin-down currents by μ-gradients must be confronted.

In these problems we need an estimate of the circulation time in the presence of μ-gradients in order to gauge the efficacy of mixing. We also need to know the penetration depth of the motions into the region with stabilizing μ-gradients. To calculate these we use here a simple model, described in § II, which permits us to isolate the most important features of the flow. In § III we give the results derived from the model. Then in § IV we discuss how such results may bear on stellar evolution calculations.

II. THE MODEL

All calculations of meridional circulation of which we are aware implicitly or explicitly make approximations which are equivalent to the Boussinesq approximation. That is, they presume that the density is nearly constant and take account of compressibility by introducing the adiabatic gradient where it is appropriate. Once results are so derived, the local density is used in their interpretation. Similarly, the neglect of the effects of curvature is common. Here, we make these approximations explicitly at the outset. We assume that the motions are slow with respect to the rotating frame, in which case the governing equations are linear.

Consider a plane-parallel layer of fluid rotating about the vertical z-axis with (constant) gravity in the negative z-direction. The vertical density gradient is taken to be stable and sufficiently large that the vertical balance of forces is hydrostatic. With these approximations, the horizontal force balance is between the Coriolis force and the pressure gradient, except where horizontal accelerations are important. We neglect viscosity in the interior of the fluid.

In this plane-parallel model we interpret x as latitude, y as longitude, and z as the radial coordinate. The (x, y, z) components of velocity are (u, v, w). We have in mind an axisymmetric situation and hence set all derivatives with respect to y equal to zero. The dynamical equations, with all these simplifications included, are

$$2\Omega v = \frac{1}{\rho_0}\frac{\partial p}{\partial x}, \quad (2.1)$$

$$\frac{\partial v}{\partial t} + 2\Omega u = 0, \quad (2.2)$$

$$\frac{\partial p}{\partial z} = -g\rho', \quad (2.3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \quad (2.4)$$

Here Ω is the angular velocity of the layer, ρ_0 is a representative density, ρ' is the density perturbation associated with the motion, and g is the gravitational acceleration (presumed constant). The quantity p is the pressure plus the centrifugal potential with the contribution from the static state subtracted out. These equations are standard in the theory of rotating stratified fluids (Veronis 1970).

Let $T_0(z)$ and $\mu_0(z)$ represent the basic stratifications of temperature and molecular weight, and T' and μ' be perturbations due to the motion. In a slowly moving fluid element, the pressure perturbation adjusts quickly, and to good approximation we have

$$\rho' = \rho_0(-\alpha_T T' + \alpha_\mu \mu'), \quad (2.5)$$

where $\alpha_T = -(\partial \ln \rho/\partial \ln T)_0$ and $\alpha_\mu = (\partial \ln \rho/\partial \ln \mu)_0$. The linearized conservation equations for T and μ are

$$\frac{\partial T'}{\partial t} + \beta_T w = Q \quad (2.6)$$

and

$$\frac{\partial \mu'}{\partial t} - \beta_\mu w = 0, \quad (2.7)$$

where

$$\beta_T = \frac{dT_0}{dz}, \qquad \beta_\mu = -\frac{d\mu_0}{dz}, \quad (2.8)$$

and Q, the heat exchange rate of the perturbation with the background, will be specified presently. Diffusion of helium is neglected. In the present considerations, α_T, α_μ, β_T, and β_μ are all positive. The model is a generalization of one used by D. W. Moore (private communication) to discuss stratified spin-down; and Moore's model is in turn an extension of a model used by Holton (1965).

The choice of boundary conditions depends somewhat on particular circumstances. The configuration we have in mind consists of a star with a μ-gradient in the core and an Eddington-Sweet circulation in the envelope. If the μ-gradient were an absolute barrier, the currents would be turned aside at the edge of the core. In that case, as Mestel (1953) has discussed, a viscous boundary layer would form at the core-envelope interface. We take this as our initial configuration and follow the subsequent development of motion in the core.

The viscous boundary layer at the edge of the core is of the Ekman type in which the principal forces are Coriolis and viscous. The detailed structure of this thin layer does not affect the interior motion; its influence is entirely through the matching condition it provides between the core and envelope velocities. Therefore, all that we need is some representation of the tendency of the envelope flow to penetrate into the core, and for this purpose a qualitatively adequate condition is the Ekman boundary condition,

$$w = -C\left[\frac{\partial v}{\partial x} - \left(\frac{\partial v}{\partial x}\right)_{\rm env}\right], \quad (2.9)$$

where C is a constant and the subscript "env" denotes evaluation at the inner edge of the envelope. The Ekman condition implies that if there is a difference in the vertical vorticity across the interface, a vertical flow follows inexorably. A number of other forms could be used, but this one is well documented, and applies to Mestel's model with its viscous boundary layer. In particular, for that case $C = \frac{1}{2}(\nu/\Omega)^{1/2}$, where ν is the kinematic viscosity. Condition (2.9) applies at $z = R_c$, where R_c is interpreted here as the core radius. At $z = 0$, corresponding loosely to the center of the star, we must have

$$w = 0 \quad \text{at} \quad z = 0. \quad (2.10)$$

In applying condition (2.9) we are neglecting initial transients whose time scale is the viscous diffusion time across the Ekman layer of thickness $\sim(\nu/\Omega)^{1/2}$. This time is of the order of one rotation period, during which the viscous layer is established and the penetrative velocity, w, is set up. These transients are not described by the equations we have just written; for the more refined calculation they entail, see the papers of Greenspan and Howard (1963) and St. Maurice and Veronis (1975). In addition, at $t = 0$ we require that T', μ', u, and v vanish.

The vorticity at the edge of the envelope in effect forces the penetration. The meridional component of the envelope velocity vanishes at the poles and equator. At the inner edge of the envelope we represent this behavior by assuming

$$v_{\rm env} = V_0 \Re(e^{ilx}), \quad (2.11)$$

where V_0 is a constant, $x = 0$ corresponds to the pole, and $x = \pi/l$ to the equator, that is, l is roughly $2R_c^{-1}$. The factor e^{ilx} will be common to all the dependent variables and $\Re$ denotes the real part.

Finally, we must specify Q. For a qualitative description of the thermal effects we can use

$$Q = -qT', \quad (2.12)$$

where q^{-1} is the thermal time of the core. The calculations in § III are performed for this case. We note, however, that we have also used the form

$$Q = \kappa\nabla^2 T' \quad (2.13)$$

to study the problem. This representation raises the order of the differential equations and complicates the mathematics. We report briefly on the details in Appendix A but simply mention here that for the large-scale effects equation (2.13) gives results similar to those found with equation (2.12) with the replacement of q by $l^2\kappa$.

III. RESPONSE OF THE CORE

After a period of adjustment, the flow in the core becomes steady in the rotating frame. The time scale of this response is the spin-up time which is one of the principal results to be derived from the model.

To solve the system of equations, we let each dependent variable vary with x like $\exp(ilx)$ and take the Laplace transform in time, giving due attention to the initial conditions. We then have a system of ordinary differential equations in z which can easily be reduced to the single equation,

$$(p+q)\frac{d^2W}{dz^2} - l^2[S_T{}^2p + S_\mu{}^2(p+q)]W = 0\,, \quad (3.1)$$

where

$$W = e^{-ilx}\int_0^\infty we^{-pt}dt\,, \quad (3.2)$$

$$S_T{}^2 = \frac{N_T{}^2}{4\Omega^2}\,, \qquad S_\mu{}^2 = \frac{N_\mu{}^2}{4\Omega^2}\,, \quad (3.3)$$

and

$$N_T{}^2 = g\alpha_T\beta_T\,, \qquad N_\mu{}^2 = g\alpha_\mu\beta_\mu\,. \quad (3.4)$$

The N's are the buoyancy frequencies due to the stratification of T and μ, while S_T and S_μ are the dimensionless stratification parameters normally used in the theory of rotating stratified fluids. As pressure will not occur in what follows, the use of p for the transform variable should not cause confusion.

Laplace transformation of the boundary conditions gives us

$$W = -ilC(V - V_0/p) \quad \text{at} \quad z = R_c\,, \quad (3.5a)$$

$$W = 0 \quad \text{at} \quad z = 0\,, \quad (3.5b)$$

where V is related to v just as W is related to w. With the help of the equations, condition (3.5a) may be rewritten as

$$pW = -C\left[2\Omega\frac{dW}{dz} - ilV_0\right] \quad \text{at} \quad z = R_c\,. \quad (3.6)$$

Note that V and W, being Laplace transforms, do not have dimensions of velocity while V_0 does.

The solution of equation (3.1) subject to conditions (3.5*b*) and (3.6) is

$$W = \frac{ilCV_0\sinh(kz)}{p\sinh(kR_c) + 2\Omega Ck\cosh(kR_c)}\,, \quad (3.7)$$

where

$$k^2 = l^2\left[\left(\frac{p}{p+q}\right)S_T{}^2 + S_\mu{}^2\right]. \quad (3.8)$$

We see that W is an even function of k, and equation (3.8) shows that p is also even in k. Hence W has no branch points in the p-plane. The Laplace inversion of W is therefore straightforward with contributions coming only from the poles of expression (3.7), and hence

$$w = ilCV_0\sum_{n=1}^\infty A_n\sinh(k_nz)\exp(p_nt + ilx)\,, \quad (3.9)$$

where

$$A_n{}^{-1} = \sinh(k_nR_c)\left\{1 + 2\Omega C\frac{d}{dp}[k\coth(kR_c)]\right\}_{k=k_n} \quad (3.10)$$

and k_n and p_n are the joint solutions of equation (3.8) with

$$p\sinh(kR_c) + 2\Omega Ck\cosh(kR_c) = 0\,. \quad (3.11)$$

The other variables of the flow can now be obtained. They are

$$u = -CV_0\sum_{n=1}^\infty A_nk_n\cosh(k_nz)\exp(p_nt + ilx)\,, \quad (3.12)$$

$$v = V_0\frac{\cosh(k_0z)}{\cosh(k_0R_c)}e^{ilx} + 2\Omega CV_0 \times \sum_{n=1}^\infty\frac{k_nA_n}{p_n}\cosh(k_nz)\exp(p_nt + ilx)\,, \quad (3.13)$$

$$\mu' = ilV_0\frac{\beta_\mu}{2\Omega}\frac{\sinh(k_0z)}{k_0\cosh(k_0R_c)}e^{ilx} + ilCV_0\beta_\mu \times \sum_{n=1}^\infty\frac{A_n}{p_n}\sinh(k_nz)\exp(p_nt + ilx)\,, \quad (3.14)$$

$$T' = -ilCV_0\beta_T\sum_{n=1}^\infty\frac{A_n}{p_n+q}\sinh(k_nz) \times \exp(p_nt + ilx)\,, \quad (3.15)$$

where

$$k_0{}^2 = l^2\lim_{p\to0}\left[\left(\frac{p}{p+q}\right)S_T{}^2 + S_\mu{}^2\right] \quad (3.16)$$

These solutions are expressed in terms of the decaying normal modes of the differential equations and boundary conditions of § II. The amplitudes A_n are determined by the initial conditions. We find that the values of p_n are all real and negative, and hence the inverses of the $|p_n|$ represent the decay times of the modes. For very large time a nonzero azimuthal velocity remains, as would be expected from the boundary conditions. The most slowly decaying mode determines the spin-up time, which is thus the largest value of $|p|^{-1}$ allowed by equations (3.8) and (3.11).

In the special case for which $q = 0$ there is only one solution for (3.8) and (3.11):

$$p = -2\Omega Ck\coth(kR_c)\,, \quad k = l(S_T{}^2 + S_\mu{}^2)^{1/2}, \quad (3.17)$$

which is essentially the result first given by Holton (1965). The core adjusts to the vorticity at its boundary in time scale of order $|p|^{-1}$, but the effects are felt only to a depth of approximately $\frac{1}{2}R_c(S_T^2 + S_\mu^2)^{-1/2}$ into the core. This case is particularly simple because the density effects due to molecular-weight and thermal effects are indistinguishable (in the present approximation) in the absence of any diffusion of either.

Another special case corresponds to that for which $S_\mu = 0$ and $q \neq 0$. In this case we can readily reduce equations (3.8) and (3.11) to

$$\tanh(kR_c) = \frac{2\Omega C(k^2 - l^2S_T^2)}{qk} \quad (3.18)$$

and

$$p = -\frac{qk^2}{k^2 - l^2S_T^2} = -2\Omega Ck \coth(kR_c) . \quad (3.19)$$

Equation (3.18) has one real root, k_1, and an infinite number of imaginary roots $k_n = i\kappa_n$ $(n = 2, 3, 4, \ldots)$. We see from equation (3.18) that $k_1^2 > l^2S_T^2$ and from equation (3.19) that $p_1 < 0$. All the other values of p are also negative. In the typical astrophysical case, the diffusion time is much longer than the dynamical time, which statement, in terms of the present model, translates into

$$\Omega ClS_T/q \gg 1 . \quad (3.20)$$

This condition simply says that the thermal time of the core is much greater than the spin-up time of a layer of thickness $\sim R_c/S_T$. In this limit $k_1 \sim lS_T$ and $p_1 \sim -2\Omega ClS_T \coth(lS_TR_c)$. With a little graphical analysis we find that $\kappa_n \sim (n - 3/2)\pi/R_c$, $0 > p_2 > p_3 > \ldots > -q$, and $p_2 \sim -\pi^2q(\pi^2 + 4R_c^2l^2S_T^2)^{-1}$. With $l = 2R_c^{-1}$ we have $|p_2| \sim \pi^2q(\pi^2 + 16S_T^2)^{-1}$, which for $S_T^2 \gg 1$ becomes $\pi^2q/(4S_T)^2$, the inverse of the Eddington-Sweet time.

These results indicate that a layer of thickness $\sim\frac{1}{2}R_c/S_T$ at the outer edge of the core adjusts quickly to the forcing at the boundary in a time $\sim R_c/(4\Omega CS_T)$. The time behavior of the rest of the core, whose scale is $R_c \approx \kappa_2^{-1}$, is qualitatively characterized by p_2, which gives the lowest decay rate of the thermal modes. Thus, the final adjustment is accomplished by a boundary-driven meridional circulation on the Eddington-Sweet time. These results have been previously derived by L. N. Howard (private communication) and D. W. Moore (private communication). For a detailed analysis of this problem see the recent paper of St. Maurice and Veronis (1975); for the effect of sidewalls see the papers of Sakurai (1969) and Walin (1968). A qualitative discussion is also available (Howard, Moore, and Spiegel 1967).

In the case of principal interest here, we have gradients of both μ and T and "diffusion" of T. The decay rates of the modes are described by equations (3.8) and (3.11) which may be rewritten as

$$\tanh(kR_c) = \frac{2\Omega Ck(k^2 - l^2S^2)}{q(k^2 - l^2S_\mu^2)} , \quad (3.21)$$

where

$$p = -\frac{q(k^2 - l^2S_\mu^2)}{k^2 - l^2S^2} = -2\Omega Ck \coth(kR_c) \quad (3.22)$$

and

$$S^2 = S_\mu^2 + S_T^2 . \quad (3.23)$$

These equations always admit one real root, k_1, and, if

$$\frac{2\Omega C}{qR_c}\frac{S^2}{S_\mu^2} < 1 , \quad (3.24)$$

there is a second real root, k_2. If this inequality is reversed, this root becomes pure imaginary. This latter case is the one that holds in astrophysical conditions where dynamical effects are more important than diffusion. At the transition between the two cases, $k = 0$ becomes a double root of equation (3.22), leading to a simple pole of equation (3.7). The remaining roots, $i\kappa_n$, are all pure imaginary. In the limit $\Omega CS^2/(qR_cS_T^2) \gg 1$ (cf. [3.20]) with $l = 2/R_c$,

$$k_1 \sim 2S/R_c , \qquad p_1 \sim -4\Omega CSR_c^{-1} \coth(2S) . \quad (3.25)$$

As in the case $S_\mu = 0$, $\kappa_n \sim (n - 3/2)\pi/R_c$ in this limit, but now $0 > p_2 > p_3 > \cdots > -q$ and

$$p_2 \sim -q\frac{\pi^2 + 16S_\mu^2}{\pi^2 + 16S^2} \quad (3.26)$$

Thus, the circulation penetrates a distance $R_c/(2S)$ into the core on the very short time scale $|p_1|^{-1}$, which is independent of the separate gradients of μ and T. The rest of the core is dynamically affected on a time scale $|p_2|^{-1}$. For $S^2 \gg 1$ and $S_\mu^2 \ll 1$, $|p_2|^{-1}$ is the Eddington-Sweet time. When $S_\mu^2 \gg 1$, the response time of the core becomes

$$|p_2|^{-1} \sim \frac{S^2}{S_\mu^2}T_{\rm KH} = \frac{N^2}{N_\mu^2}T_{\rm KH} , \quad (3.27)$$

where $T_{\rm KH}$ is the thermal time of the core (approximated by q^{-1}) and

$$N^2 = N_\mu^2 + N_T^2 . \quad (3.28)$$

We must now determine how much of the core is susceptible to these effects of circulation. For this we return to the solutions expressed in equations (3.12)–(3.16) and look at them for times large compared to $|p_2|^{-1}$. We see that the spatial dependence for large time is characterized entirely by k_0^{-1} which gives the penetration depth of the secondary circulation. This depth is

$$h \equiv k_0^{-1} = \frac{R_c}{2S_\mu} . \quad (3.29)$$

It is this quantity that determines the softness of the μ-barrier, since a meridional current exterior to the core will penetrate a distance k_0^{-1} into the core. If we introduce expressions (1.8), (3.3), and (3.4) into equation (3.29), we find that

$$h = \frac{R_c \Omega}{(g|d \ln \mu_0/dz|^{1/2})}, \quad (3.30)$$

where we have replaced α_μ by μ_0^{-1}.

IV. DISCUSSION

The results just presented bear on the penetration of a μ-barrier by meridional currents. Evidently a number of approximations have been made, but for the most part these are just explicit forms of approximations that are implicitly made in most treatments of stellar circulation theory. An important omission made here that is not customary in such treatments is the neglect of meridional currents driven by horizontal gradients of μ; however, such currents are qualitatively similar to the ones studied here. For both kinds of currents we must attempt to confront the question raised by Mestel (1953): Can a μ-barrier be set up despite rotational mixing? The results found here suggest that such a barrier can be penetrated and we must allow for some mixing. Once we do this, changes in N_μ will occur and these will affect the circulation times. Thus, on the time scale of stellar evolution the mixing problem becomes nonlinear and the present results cannot be applied directly. They do, however, give us some guide to what happens, and we are led to the following scheme for including the effects of rotational mixing in calculations of stellar evolution.

Consider a rotating star which has just arrived on the early main sequence. The star has a convective core whose interaction with the circulation further complicates matters. Fortunately, we have the simplification that convective mixing proceeds much more rapidly than rotational mixing. The direct interaction between convection and circulation probably occurs outside the unstable core itself, that is, beyond the radius at which β_T, the difference between the temperature gradient and the adiabatic gradient, vanishes. Therefore we must know how far the convective motions overshoot the radius at which $\beta_T = 0$.

We suggest that an estimate for the distance of convective overshoot, though normally difficult to obtain, may be simply made for a rotating star, based on the discussion of § III. There we saw that a flow impinging on the boundary of a rotating, stably stratified fluid will penetrate a distance into the core which is independent of the incident speed, provided it is not too large. The cause of this penetration is the rigidity imposed on the fluid by rotation. If there were no stratification, the Taylor-Proudman theorem (Veronis 1970) would hold (in our simple model) and the motion would be independent of z away from the boundary layers. The stable stratification prevents the motion from being completely independent of z, though there is a finite distance over which the rotation can assert itself and give an effective rigidity to the fluid. For qualitative purposes, it should not matter whether the origin of a penetrative velocity is Ekman pumping, as in equation (1.9), or a convective motion. On a time scale short compared to the thermal time, the convection will overshoot at least to a distance implied by equation (3.16). If we consider the largest convective scales, the horizontal scale $l^{-1} \sim R_c$. The distance of overshooting is then $\sim R_c\Omega/N$ (or even more if the convection were sufficiently vigorous).

If we evolve the star over a time short compared to the circulation time, we can estimate the new μ-distribution assuming that it was known at the outset. In this new distribution we assume that the core is homogeneous out to the radius where $\beta_T = 0$ and that thereafter the value of μ drops linearly to its initial value in a distance $R_c N/\Omega$, where N is evaluated for the initial stratification. This procedure can be continued over several time steps until the material extruded by the core is swept away by the meridional circulation. How do we then allow for this latter effect?

Let us neglect the influence of the overshooting on the temperature distribution as well as the possible turbulent conductivity and viscosity it provides. Then the circulation behaves qualitatively as described in § III when it interacts with the newly established μ-gradient. As long as the region of μ-gradient is less thick than h of equation (3.30), the currents should successfully mix the material outward on the time scale indicated by equation (3.26). Thus, the important length scale is

$$h = R_c \Omega / N_\mu ;$$

and as long as the region of μ-gradient is less thick than this, mixing between core and envelope takes place on the time scale

$$\tau = \frac{\pi^2 \Omega^2 + 4N^2}{\pi^2 \Omega^2 + 4N_\mu^2} \tau_c ,$$

where τ_c is the thermal time scale of the core. These formulae are further discussed in Appendix B, where we find that until the μ-gradient is large enough that $N_\mu > \Omega$, it cannot seriously impede mixing.

As evolution proceeds, with mixing included as described here, it is still possible that a sufficiently large value of μ builds up outside the convective core so that the criterion for inhibition of mixing by the μ-gradient is met. But mixing of the core does not stop completely even then. Rather, a layer of thickness h at the top of the barrier is continually eroded away, and this should be allowed for. The formulae indicate how the rates of extrusion from the convective core and of erosion of the μ-barrier can be followed.

While this procedure provides ways to estimate the rate of exchange between core and evelope, it does not tell us how the matter from the core is to be distributed in the envelope. On the basis of analogous problems in convection theory and of observations of salt layers in oceans (Turner 1973) we conjecture that a series of μ-layers would form in the envelope. It would be necessary to study the transport of μ through these layers, and to deal with this we would need to understand the structures of the layer interfaces. Instead, we propose in first approximation that one simply mix all the material from the core uniformly through the envelope. Such a proposal evidently involves a great deal of uncertainty, and it is but one of the difficulties which have been sidestepped here. We have equally neglected the formation of horizontal μ-gradients in the overshooting. Moreover, effects of evolution, such as the shrinking of the core and the spin-up currents resulting from its enhanced angular velocity, may play a vital role. And the results obtained in § III are themselves uncertain, the model being so primitive. But the scheme does provide some estimate of the amount of mixing, and we feel that something of this kind should be tried in a stellar evolution calculation, if only to test systematically the consequences of an effect whose neglect seems to us to be an even cruder approximation than that proposed here.

We are indepted to the NSF for support under grant MPS 75-05660, to the John Simon Guggenheim Foundation for a fellowship, to the 1975 GFD Summer School at WHOI for hospitality during the preparation of the manuscript, to Dr. G. Vauclaire for a discussion of evolutionary problems, and to Professor D. W. Moore for his comments on the manuscript.

APPENDIX A

In this Appendix we examine the consequences of using the radiative diffusion representation (2.13) for the heat exchange rate Q. Inserting equation (2.13) into equation (2.6) and proceeding as before, we again obtain a single equation for the vertical velocity, W, which is

$$\left[p - \kappa\left(\frac{d^2}{dz^2} - l^2\right)\right]\frac{d^2W}{dz^2} - l^2\left\{S_T{}^2p + S_\mu{}^2\left[p - \kappa\left(\frac{d^2}{dz^2} - l^2\right)\right]\right\}W = 0\,. \tag{A1}$$

In contrast to equation (3.1), the governing differential equation (A1) is fourth order and thus four boundary conditions are required. Two of these are the conditions (3.5b) and (3.6) used in the text:

$$W = 0 \quad \text{at} \quad z = 0 \tag{A2}$$

and

$$pW = -C\left(2\Omega\frac{dW}{dz} - ilV_0\right) \quad \text{at} \quad z = R_c\,, \tag{A3}$$

while the other two are derived from conditions on the temperature. We assume first that zero thermal flux emanates from the center of the core and hence

$$\frac{dT'}{dz} = 0 \quad \text{at} \quad z = 0\,. \tag{A4}$$

The condition at the edge of the core is more subtle and requires a matching to the envelope. However, it will be of the form

$$T' + \eta\frac{dT'}{dz} = 0 \quad \text{at} \quad z = R_c\,. \tag{A5}$$

Though η is actually a functional of the flow in the envelope, we treat it as a given constant.

The general solution of equation (A1) is

$$W = c_1 \sinh k_1z + c_2 \sinh k_2z + c_2 \cosh k_1z + c_4 \cosh k_2z\,, \tag{A6}$$

where $k_1{}^2$ and $k_2{}^2$ are the solutions of

$$k^4 - [l^2(1 + S_\mu{}^2) + (p/\kappa)]k^2 + [(S^2p/\kappa) + l^2S_\mu{}^2]l^2 = 0\,. \tag{A7}$$

From equation (2.6) with Q given by equation (2.13), we can write the Laplace-transformed thermal field,

$$\theta = e^{-ilx}\int_0^\infty T'e^{-pt}dt\,, \tag{A8}$$

as

$$\theta = -\beta_T/(l^2S_T{}^2p)[(k_1{}^2 - l^2S_\mu{}^2)(c_1 \sinh k_1z + c_3 \cosh k_1z) + (k_2{}^2 - l^2S_\mu{}^2)(c_2 \sinh k_2z + c_4 \cosh k_2z)]\,. \tag{A9}$$

In principle, the method of solution is now straightforward: evaluate the four constants c_1, c_2, c_3, c_4 from the inhomogeneous boundary conditions (A2)–(A5) and then determine the inverse Laplace transform of W to obtain

w and thence the remaining fields. In practice the Laplace inversion is extremely cumbersome and could not be expressed in terms of known functions. We therefore content ourselves with the determination of the large-time solutions and an estimate of the time scale for the most slowly decaying mode.

The large-time solution is obtained by considering the limit $p \to 0$. In this limit equation (A7) becomes

$$k^4 - l^2(1 + S_\mu{}^2)k^2 + l^4 S_\mu{}^2 = 0\,, \tag{A10}$$

with solutions $k_1 = lS_\mu$ and $k_2 = l$. On substituting these values into equations (A6) and (A9) and solving the resulting four equations for c_1 to c_4, we find that

$$v \sim V_0 \frac{\cosh lS_\mu z}{\cosh lS_\mu R_c} e^{ilx} \quad \text{as} \quad t \to \infty \tag{A11}$$

and

$$\mu' \sim \frac{iV_0\beta_\mu}{2\Omega S_\mu} \frac{\sinh lS_\mu z}{\cosh lS_\mu R_c} e^{ilx} \quad \text{as} \quad t \to \infty\,, \tag{A12}$$

while the u, w and T' fields tend to zero in this limit. These results are exactly the same as those obtained in § III (cf. eqs. [3.12]–[3.16]) using the representation (2.12) for Q.

The time scale of the decaying portions of the solution emerges from equation (A7), which with a slight arrangement can be written as

$$p = \frac{\kappa(k^2 - l^2)(k^2 - l^2 S_\mu{}^2)}{k^2 - l^2 S^2}\,. \tag{A13}$$

The exact evaluation of the eigenvalues, k^2, requires determination of the full solution as discussed above. However, without carrying out the solution in detail we can see that the eigenvalue leading to the smallest value of p will be of the form $k = i\pi\zeta/(2R_c)$, where ζ is a constant of order unity. Numerical evaluation for a number of values of S_t, S_μ, C, and η confirms that this is true; and to good approximation we have

$$p = -\frac{\kappa}{4R_c{}^2}(\pi^2\zeta^2 + 16)\frac{(\pi^2\zeta^2 + 16S_\mu{}^2)}{(\pi^2\zeta^2 + 16S^2)}\,, \tag{A14}$$

where we have replaced l by $2/R_c$. Equation (A14) is to be compared with equation (3.26); after replacement of q by $\kappa(\pi^2\zeta^2 + 16)/(4R_c{}^2)$ the two equations differ only by numerical factors. When $S_\mu{}^2 \gg 1$, the response time becomes

$$|p|^{-1} \sim \frac{S^2}{S_\mu{}^2} T_{\rm KH}\,, \tag{A15}$$

where the thermal time of the core, $T_{\rm KH}$, is here approximated by $[4R_c{}^2/(\pi^2\zeta^2 + 16)]\kappa^{-1}$. Equation (A15) is identical to equation (3.27).

This analysis deals only with the decay time of the longest-lived normal mode. We will not discuss here the new modes that enter when the radiative terms are described by the diffusion approximation. Some of those modes are mainly thermal. They may be important for stellar evolution since they might affect burning rates. This question needs a fuller investigation, but for that a more realistic model should be devised.

APPENDIX B

Kippenhahn (1974) estimates the speed of circulation in a region with a μ-gradient as

$$v_\mu \approx \frac{q}{\alpha_T \beta_T \mu} |\mu'|\,, \tag{B1}$$

where μ' is the perturbation associated with the motion and we have converted to our notation and neglected deviations from ideal gas behavior. To obtain a result which may be compared with ours we must specify μ'. If we set $\mu' \approx \beta_\mu R_c$, the circulation time, R_c/v_μ, becomes

$$\tau_\mu = \frac{N_T{}^2}{N_\mu{}^2} q_c{}^{-1}. \tag{B2}$$

The quantity q_c^{-1} is identified with the Kelvin-Helmholtz time of the core if, as we assume now, the horizontal scale of motion $l^{-1} \sim R_c$. On the other hand, from expression (3.26) we find the circulation time

$$\tau = \frac{\pi^2\Omega^2 + 4N^2}{\pi^2\Omega^2 + 4N_\mu^2} q_c^{-1} . \tag{B3}$$

When $N_T^2 \gg N_\mu^2 \gg \Omega^2$, we recover equation (B2). But there is a disquieting feature since we would expect to recover the correct time scale for meridional circulation when $N_\mu \to 0$, and this is not possible with equation (B2). Perhaps another choice of μ' should be introduced into equation (B1); there is one suggested in § 2.3 of Kippenhahn's paper, but that does not seem to alleviate the problem. The consequence of the difference between the times expressed in equations (B2) and (B3) appears in the discussion of the penetration of a μ-barrier by Eddington-Sweet circulation. Kippenhahn suggests as a qualitative criterion for the inhibition of such penetration that $v_\mu > v_{\rm E}$, where $v_{\rm E}$ is the Eddington-Sweet circulation speed. In our notation, his expression for the Eddington-Sweet time $R/v_{\rm E}$ is

$$\tau_{\rm E} = \frac{H_p}{R}\frac{N_T^2}{\Omega^2} q^{-1} , \tag{B4}$$

where H_p is the pressure scale height and q^{-1} is the Kelvin-Helmholtz time of the star. His criterion becomes $\tau_\mu/\tau_{\rm E} < R_c/R$ when expressed in terms of the time scales. On the other hand, from the appropriate special case considered in § III, or immediately from equation (B3) when $N_\mu = 0$, we see that

$$\tau_0 = \frac{\pi^2\Omega^2 + 4N_T^2}{\pi^2\Omega^2} q^{-1} . \tag{B5}$$

should be used in such comparisons and not expression (B4). The difference between τ_0 and $\tau_{\rm E}$ is that the latter holds only for $\Omega^2 \ll N_T^2$. Now if we use (B3) and (B5) instead of (B2) and (B4) in Kippenhahn's criterion, we find, using the notation of equation (3.3),

$$\frac{1 + S_T^2 + S_\mu^2}{(1 + S_T^2)(1 + S_\mu^2)} < \left(\frac{R_c}{R}\right)\left(\frac{q_c}{q}\right) . \tag{B6}$$

Since q (or q_c) varies inversely as R (or R_c), the right-hand side is greater than unity and this criterion for prevention of mixing is always met. This is, as it were, built into expression (B3).

In fact, as we saw in the previous section, the envelope currents always penetrate to a finite depth into the core,

$$h = R_c\Omega/N_\mu , \tag{B7}$$

and this corresponds to the transition layer mentioned by Kippenhahn. But this formula makes it clear that even if the rotation is small in terms of the total density stratification, its effects are central. For until the μ-gradient builds to the point where

$$N_\mu > \Omega , \tag{B8}$$

the μ-barrier is totally ineffective. And when condition (4.8) is met, the μ-barrier can stop the circulation only after it has penetrated to a depth h in the core. Therefore, condition (B8) would seem to be the appropriate rule of thumb for the onset of an effective μ-barrier.

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