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Pouring viscous fluid out of a tipped container in minimal time

Daisuke Takagi* and Herbert E. Huppert
Institute of Theoretical Geophysics, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

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Emptying a container partially filled with viscous fluid can be a frustratingly slow process. It takes time for the fluid to even begin discharging after tipping the container to develop a draining film on the interior surface. To study the effects of the shape and the tipping angle of the container, we predict the time required for the fluid to begin discharging in two simple geometries. In addition, the volume of the fluid yet to be discharged at subsequent times is predicted to decrease as $t^{-1/2}$ or t^{-1} for flow driven along a plane or corner, respectively. We compare these theoretical predictions with laboratory experiments and discuss how viscous fluids could be poured out most effectively.

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Viscous fluids in a wide range of contexts are stored and transported in containers. Bottles of shampoo, buckets of detergent, cans of paint, jars of honey, and tanks of oil represent familiar examples in the cosmetic, cleaning, coating, food, and petroleum industries, respectively. A common method of retrieving the fluid is by simply tipping the container. However, the process can be slow, particularly when there is little fluid remaining in the container due to the development of a thin film that drains slowly on the interior surface. The motivation of the present study is to investigate how viscous fluid can be poured out most quickly from a tipped container, a problem that is yet to be resolved despite its importance in many applications. There are practical benefits of minimizing the time required to retrieve the first drip of fluid after tipping. In addition, there are economic advantages to minimizing the fluid remaining in the container, which is usually disposed of as waste.

To gain some physical insight, we consider a simplified situation wherein a thin film of Newtonian fluid develops and flows at low Reynolds number. We develop a mathematical model using lubrication theory, as adopted successfully for other relevant situations concerning the removal of fluid from a rigid surface [1–6], including the pioneering studies of drainage on a vertical plate [4], the scraping of fluid on a plane surface [5], and the film that is left behind as a viscous fluid is blown from an open-ended tube [6]. We obtain analytical solutions predicting the time needed to start pouring and the fluid volume yet to be discharged at subsequent times in two simple geometries and test the predictions against a series of experiments conducted in the laboratory.

Consider a rectangular container of vertical height H, width W, and breadth B, partially filled with Newtonian fluid of volume $V_0 \ll HWB$, dynamic viscosity μ, and density ρ. A Cartesian coordinate system is fixed in the container with the origin representing a corner on the base, the x axis initially points vertically, and the y and z axes extend horizontally along the breadth and width, respectively. After the container has been tipped at an angle θ beyond the horizontal and held fixed, the fluid is driven by gravity and eventually pours out of the container, as sketched in Fig. 1(a). The initial flow inside the container is primarily in the direction of the $-z$ axis and is then followed by motion in the x axis, assuming that effects due to walls confining the current and the upstream end of the flow on the base are negligible.

When the fluid thickness is much smaller than the width B or the lengths W and H of the flow, the volumetric flow rate per unit cross-stream width is given by $q = g_* h^3/3\nu$, where g_* is the component of gravity g along the flow, h is the flow thickness, and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid [7]. This holds provided effects due to fluid inertia and surface tension are neglected in the dynamical regime of low Reynolds number $\mathrm{Re} = UL/\nu$ and high Bond number $\mathrm{Bo} = \rho g L^2/\sigma$, where σ is the surface tension, $U = q/h$ is a characteristic scale for the velocity, and L is a suitable length scale of the flow. The relevant dimensionless parameters are

$$\mathrm{Re} = \frac{g V_0^3}{W^3 B^3 \nu^2} \ll 1, \quad \mathrm{Bo} = \frac{\rho g B^2}{\sigma} \gg 1, \tag{1}$$

where V_0/WB is the initial fluid thickness and sets the length scale relevant for the Reynolds number Re, while B is the length scale relevant for the Bond number Bo.

The local conservation of mass in the $-z$ direction, $\partial h/\partial t = \partial q/\partial z$, is solved subject to the approximate condition that the film thickness is negligible after the fluid has drained sufficiently at $z = W$, where we set $h = 0$. The solution for the fluid thickness on the base of the container,

$$h_b(z,t) = \left(\frac{\nu(W - z)}{g \cos\theta\, t}\right)^{1/2}, \tag{2}$$

is obtained by the method of characteristics [8]. The solution in Eq. (2) holds provided the base of the container does not deviate much from the vertical to allow the fluid to drain without dripping [9]. The fluid thickness along the x axis,

$$h(x,t) = \left(\frac{\nu(x + W \tan^{1/3}\theta)}{g \sin\theta\, t}\right)^{1/2}, \tag{3}$$

is obtained by conserving mass in the x direction, $\partial h/\partial t = -\partial q/\partial x$, subject to the condition that the flow rate is continuous in the limits as $z \to 0$ and $x \to 0$. Thus the flux of fluid down the $-z$ axis at the origin is converted into the flux

*Present address: Courant Institute of Mathematical Sciences, New York University, New York, New York 10012, USA. takagi@cims.nyu.edu

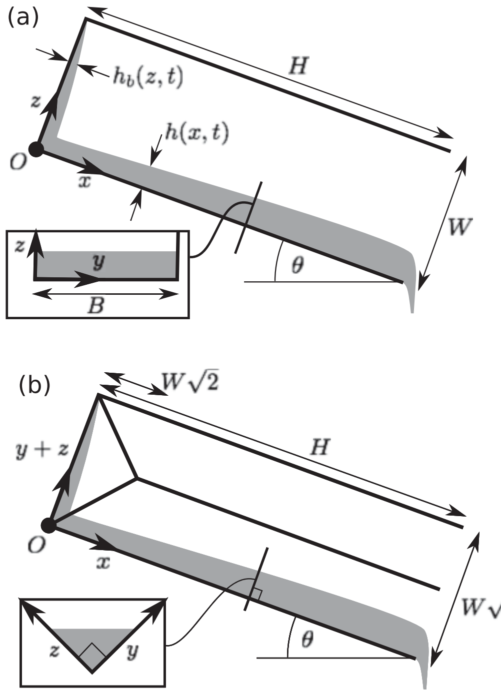

FIG. 1. Sketch of (a) a tipped container of rectangular cross section such that the bulk flow is along two planes and (b) a square container with a wedged base such that the bulk flow is along the wedge and a corner of the container. Each inset shows a cross section of the flow.

in the x direction and this becomes a boundary condition. This condition determines the constant of integration and gives rise to the term $W \tan^{1/3}\theta$ in Eq. (3). The region at the corner, where the flow is two dimensional and possibly influenced by capillary effects, is small based on the assumption that the film thickness is much smaller than any length scale of the container. The neglect of this small region is justified by the agreement between the theory and the experiments, as reported below. Note that the fluid thickness along the x axis is equivalent to that on an inclined plane where the fluid is released at a virtual origin at $x = -W\tan^{1/3}\theta$.

The front of the current remains stable in the container as long as it travels a distance less than a critical length proportional to $(V_0/B)^{1/2}$ [10]. In that case, the front of the current reaches the open end of the container at time $\bar{T}$, which is determined by the condition $\int_0^H h(x,\bar{T})\,dx + \int_0^W h_b(z,\bar{T})\,dz = V_0/B$. By substituting Eqs. (2) and (3) into this condition and rearranging, we determine the fluid to begin pouring out of a flat edge after a dimensionless time

$$\bar{T}\left(\frac{\nu B^2 H^{3/2} W^{3/2}}{g V_0^2}\right)^{-1} = \frac{4}{9\sin\theta}(a^{1/2} + a^{-1/2}\tan^{1/3}\theta)^3, \tag{4}$$

where $a = H/W$ is the aspect ratio of the container. The time $\bar{T}$ scales like ϵ^{-2}, where $\epsilon \ll 1$ is defined as the proportion of the container initially occupied by the viscous fluid V_0/V_c, with $V_c = WBH$. The expression in parentheses on the left-hand side of Eq. (4) is a time scale of the problem and decreases in value by exchanging the values of B and W if $B > W$, indicating that the discharge of fluid from a given container is enhanced by directing the flow on a plane of minimal area. For a given container of aspect ratio a, the dimensionless function on the right-hand side of Eq. (4) attains a minimum at $\theta = \arctan(a^{3/7})$. A global minimum of $2^{11/2}/3^2$ is attained at $a = 1$ and $\theta = \pi/4$, indicating that the fluid is retrieved most rapidly, with all quantities other than a and θ fixed, from a container of identical height and width and tipping it 45° to the horizontal.

Once the fluid has started to pour out, the volume of the fluid remaining on the interior planes of the container, $\bar{V} = B\int_0^H h(x,t)\,dx + B\int_0^W h_b(z,t)\,dz$, can be expressed as

$$\bar{V} = V_0(t/\bar{T})^{-1/2}, \tag{5}$$

using Eqs. (2)–(4). Note that $\bar{T} \sim V_0^{-2}$ from Eq. (4), so the functions $\bar{V}(t)$ for different V_0 fall on a curve, independent of V_0 for times $t > \bar{T}$. The volume of the fluid yet to be discharged scales like the flow thickness and evolves with time like $t^{-1/2}$ because the characteristic width B of the flow remains constant. A different scaling relationship between the volume and time arises when the width, in addition to the thickness of the flow, evolves in time, as shown next in a container of different shape.

Consider now a configuration where the fluid drains from a container such that the bulk flow is along a wedge. This can be set up in a square container of vertical height H and width W by inserting a right-angled wedge along a diagonal of the base. Figure 1(b) shows a sketch of the flow along two successive channels of a V-shaped cross section, which arises after tipping the tank initially filled with a small volume of fluid V_0. The free surface remains approximately flat across the flow as the fluid depth declines in time provided the contact line recedes over a time scale shorter than the time scale associated with the main flow. The volumetric flow rate along a wedge is given by $Q_x = K g_* h^4/\nu$, where K is a dimensionless function of the angle at the vertex that attains a maximum of $K \approx 0.68$ when the wedge is right angled [11]. The Reynolds and Bond numbers are modified accordingly by setting the relevant length scale to $(V_0/W)^{1/2}$, which is proportional to the initial thickness of the fluid.

The fluid begins to pour out of a corner at dimensionless time

$$\check{T}\left(\frac{\nu H W}{g V_0}\right)^{-1} = \frac{1}{4K\sin\theta}(a^{1/2} + a^{-1/2}\sqrt{2}\tan^{1/2}\theta)^2, \tag{6}$$

which is obtained using the same methods as before. The time $\check{T}$ required for the fluid to begin discharging after tipping the container scales like ϵ^{-1}, where $\epsilon = V_0/V_c$ with $V_c = W^2(H - W/3\sqrt{2})$. The scaling ϵ^{-1} is greater than ϵ^{-2} for small ϵ, indicating that small volumes of fluid start to discharge sooner out of a corner than a flat edge of a rectangular container of similar dimensions. The volume of the fluid remaining inside corners is either V_0 for $t \leqslant \check{T}$ or given by

$$\check{V} = V_0(t/\check{T})^{-1} \tag{7}$$

for $t > \check{T}$. At large times, the volume of the fluid remaining along the corners decreases more rapidly than that along the flat edges of a rectangular container.

We conducted experiments to test the theoretical predictions of the time required for the fluid to begin discharging and the subsequent decline in the volume of the fluid remaining inside different containers. An acrylic glass tank of height $H = 300$ mm with a square cross section of width $W = 100$ mm was initially vertical and partially filled with different volumes of either glycerine of viscosity $\nu = 890 \pm 30\ \mathrm{mm}^2/\mathrm{s}$ or golden syrup of viscosity $\nu = 38\,000 \pm 8000\ \mathrm{mm}^2/\mathrm{s}$, the uncertainty attributed to fluctuations in room temperature $20.5 \pm 1\,^\circ\mathrm{C}$. The viscosities were measured using U-tube viscometers. A right-angled corner was constructed along a diagonal of the base by inserting a pair of aluminum plates and sealing any gaps with tape. The tank was rapidly tipped at either $\theta = 5^\circ$ or 35° to the horizontal and held fixed on clamp stands. Any fluid discharged from the tipped container was collected inside a beaker on a weight balance. The mass of the discharged fluid was recorded to an accuracy of 0.1 g approximately every one-third of a second.

Figure 2(a) shows the interval t^* between the moment when the container is rapidly tipped and when the fluid starts discharging from a flat edge of the tank, as a function of the proportion of the container initially occupied by the fluid, $\epsilon = V_0/V_c$, where $V_c = W^2 H$. The interval is rescaled by $\bar{T}\epsilon^2$, which is independent of the initial volume of fluid in the container. Figure 2(b) shows the corresponding plot of the time, rescaled by $\check{T}\epsilon$, required for the fluid to begin discharging from a corner of the tank with a wedged base, where $V_c = W^2(H - W/3\sqrt{2})$. In both cases, the experimental data agree with the theoretical predictions of the dimensionless times, which scale like ϵ^{-1} and ϵ^{-2}, depending on the geometry.

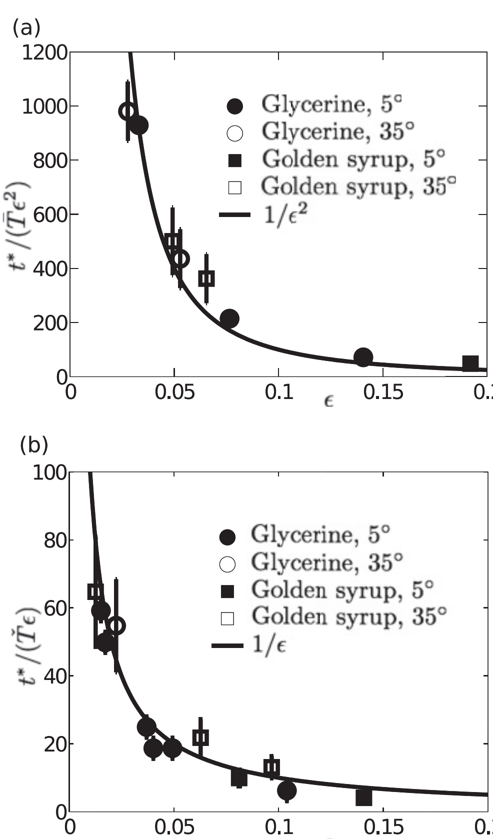

FIG. 2. Rescaled time when fluid begins to pour out of (a) a flat edge of a square container and (b) a corner of a square container with a wedged base, plotted against the proportion of the container initially occupied by the fluid.

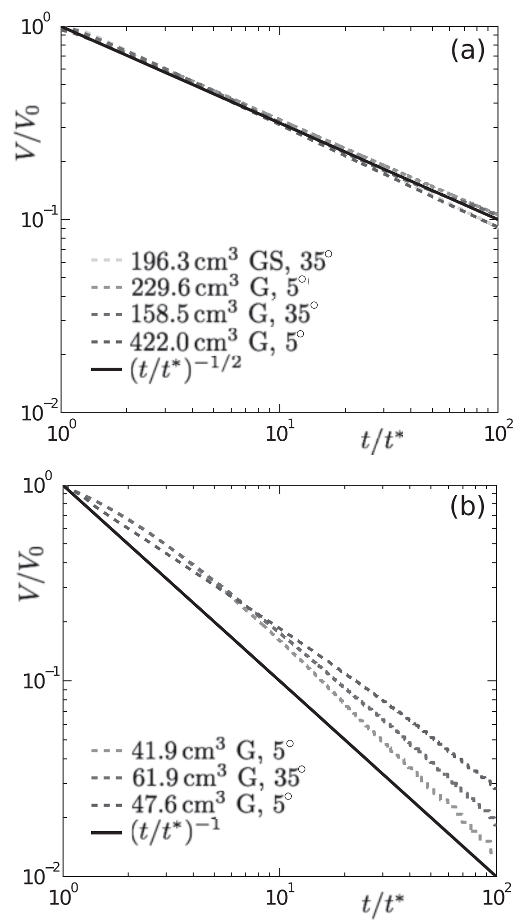

FIG. 3. Proportion of glycerine (G) and golden syrup (GS) remaining along (a) flat edges of a square container and (b) corners of a square container with a wedged base, plotted on logarithmic scales against rescaled time.

Figure 3(a) shows the proportion of fluid discharged out of a flat edge in different experiments, conducted by varying the tipping angle, the initial volume, and the viscosity of the fluid. The fluid pours out in the form of threads and then droplets, which are not incorporated in our simplified model. However, the excellent agreement between the theoretical curve and the experimental data indicate that the threads and droplets do not influence the bulk flow inside the container. The collapse of the experimental data onto the theoretical line of the form in Eq. (5) is also obtained with golden syrup in different containers, including an acrylic glass tank of circular cross section and along a corner of a square tank, with and without a wedge on the base. The volume of golden syrup decreases like $t^{-1/2}$ rather than t^{-1} along corners, possibly because the characteristic width of the flow remains constant. This is attributed to the contact line that appears pinned on the acrylic glass over the time scale of the main flow. The contact line does not recede and maintains a flat free surface across the flow, as assumed in the theory.

Figure 3(b) shows the remaining proportion of glycerine inside a corner of a square tank with a wedged base. The experimental data do not collapse as well onto the theoretical curve, possibly due to effects of the receding contact line and surface tension that are not incorporated into the model, but which could be studied in the future. Nevertheless, the remaining volume of glycerine flowing along a corner scales approximately like t^{-1}, as predicted, which declines more rapidly than $t^{-1/2}$ on a flat edge of the square container.

To gain a better understanding of the rapid discharge from a corner rather than a flat edge, we consider the cross-sectional area of the flow divided by the wetted perimeter as a measure of the efficiency of the flow. The wetted perimeter of the flow down a rectangular channel approaches the width of the channel as the fluid drains. In contrast, the wetted perimeter of the flow down a V-shaped channel is $2\sqrt{2A/\sin\beta}$, where A is the cross-sectional area of the flow and β is the internal angle at the vertex of the channel. As A decreases sufficiently with time, the resistive forces acting on the wetted perimeter are minimized by setting $\beta = \pi/2$. This indicates that a right-angled wedge, among the other shapes considered here, is the optimal shape of the channel that allows fluid of ever-thinning depth to be transported most efficiently. This holds provided the free surface of the flow is flat across the channel, as expected when the contact line recedes rapidly.

Considerable variations in the elevation of the free surface may arise across the channel, for example, due to effects of contact line pinning on the sides of the channel. The effects of the contact line on the bulk flow down a wide rectangular channel are negligible. However, the effects may significantly influence the bulk structure of the flow down a wedge. This is consistent with the observation that the agreement between the simplified theory and the experiments for flow down a rectangular channel is better than for flow along a wedge.

We conclude that the discharge of viscous fluid is influenced considerably by the shape and the tipping angle of the container. The discharge rate can be enhanced by using a container of identical height and width and tipping it 45° beyond the horizontal. The current study of pouring Newtonian fluid could be extended to incorporate additional effects in the future, such as the non-Newtonian properties of the flowing material. Other effects include the formation of threads and drops as a result of dripping inside the container, a time-dependent angle of tipping, and an oscillatory force that is applied by shaking the container.

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