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Some remarks on the initiation of inertial Taylor columns

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The quasi-geostrophic flow over an obstacle placed on the lower of two horizontal planes in rapid rotation about the vertical z axis is considered. The flow field is calculated in the limit of vanishing viscosity after assuming the flow far upstream of the obstacle to be uniform, of magnitude V. The effects that occur in homogeneous flow are compared with those that occur in stratified flow.

If the flow is homogeneous, there is a region of closed streamlines if

$$h_0 R^{-1} > \min_r \left[r \Big/ \int_0^r x h(x)\, dx \right], \tag{1}$$

where the obstacle is assumed to be cylindrically symmetric and given by $z = Hh_0 h(r/L)$, H is the distance between the planes and R is the Rossby number $V/(fL)$. For any obstacle the right-hand side of (1) is greater than zero and hence h_0 must be positive for a closed-streamline region to occur. It is argued, and illustrated by a particular example, that because (1) involves an integral of $h(x)$ a representative flow pattern can be obtained for obstacles of less than critical height by considering the special case of a flat-topped obstacle, as is done by Ingersoll (1969).

If the flow is stratified with constant Brunt–Väisälä frequency N, the condition for the existence of closed streamlines is shown to be

$$h_0 R^{-1} > \min_r \left[B \int_0^\infty \int_0^\infty xt \coth(Bt)\, h(x)\, J_0(tx)\, J_1(tr)\, dt\, dx \right]^{-1}, \tag{2}$$

where $B = NH/fL$. In contrast to the homogeneous situation, the right-hand side of (2) can be zero and is so if the obstacle is somewhere vertical. Such obstacles will produce a closed-streamline region no matter now small their height and will hence *not* lead to patterns representative of smooth obstacles. This is because a stratified column of fluid cannot be stretched or compressed over an infinitesimal distance. Instead, the column bends markedly and the fluid flows around the obstacle. The critical conditions (1) and (2) for a number of specific obstacles are calculated and discussed.

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1. Introduction

Slow steady inviscid flow in a rapidly rotating homogeneous system is independent of the co-ordinates parallel to the rotation axis; this is the well-known Taylor–Proudman theorem (Proudman 1916; Taylor 1917). Taylor (1923) wondered what the implications of this theorem would be for flow between two rigid planes perpendicular to the vertical rotation axis with an obstacle placed on one of the planes. In a beautiful experiment he showed that although "this idea appears fantastic" the fluid in the vertical cylinder circumscribing the obstacle was stagnant with respect to the obstacle and the flow consisted of vertical columns of fluid outside the stagnant region moving without changing their length.

Interest in this concept was increased when Hide (1961) conjectured that Jupiter's Great Red Spot was a manifestation of a Taylor column, a term Hide coined for the stagnant region above the obstacle. Hide's suggestion aroused considerable controversy and many investigations have been undertaken aiming to prove (e.g. Hide & Ibbetson 1966, 1968; Ingersoll 1969) or disprove (e.g. Stone & Baker 1968; Goody 1969) his conjecture. The investigation is still incomplete, and the existence of a Taylor column on Jupiter remains no more, nor less, than a conjecture. The difficulties in proving, or rejecting, the conjecture are twofold. First, little data is available concerning either the Jovian surface or the Jovian atmosphere. Thus we know neither the shape of the 'obstacle' nor the properties or flow characteristics of the fluid in which it is imbedded. Second, Taylor's work has not been extended sufficiently, so that we do not yet know the solution to the relevant, purely theoretical problem; that is, the form of motion to be expected in an arbitrarily stratified, slightly viscous, shear flow over prescribed topography in a rapidly rotating system incorporating the β-effect.

It should be here emphasized that the solution of this theoretical problem will have important applications independent of its relevance to the Great Red Spot since Taylor columns possibly exist in other geophysical contexts: they may be prevalent in sections of the world's oceans and occur in the atmospheres of other planets. For example, Hogg (1973) has argued that the large region of comparatively dense water observed by Swallow & Caston (1973) in the Northwest Mediterranean during the MEDOC cruise is a two-dimensional Taylor column formed by the Rhone Deep Sea Fan, which stands out of the abyssal plane in that area.

The purpose of this note is to make some remarks on the theory of the initiation of steady Taylor columns in a homogeneous and in a stratified system. The system we consider consists of two horizontal planes in uniform rotation about a vertical axis. An obstacle of everywhere non-negative height is attached to the lower plane and the fluid is constrained to flow between the two planes. We assume that the volume of the obstacle is finite and that the flow far upstream of the obstacle is uniform. Only obstacles which are circularly symmetric are considered, which simplifies the resulting analysis by allowing the solution to be expressed in polar co-ordinates. To calculate the flow due to an obstacle of arbitrary shape Cartesian co-ordinates can be used in a manner almost identical to that below with similar results.

We first investigate (§2) the effects in a homogeneous flow using the same approach as Ingersoll (1969) though our results are given in greater generality. In particular, equation (2.23), the condition for the initiation of a Taylor column by a circularly symmetric obstacle, is new. From this equation we show that in a homogeneous flow all obstacles must be of non-zero height in order to induce a Taylor column. We then (§3) investigate the flow in a stratified system, in a manner paralleling that of §2. The effects of stratification have been previously investigated by Hogg (1973) by a technique applicable only for a flat-topped cylinder. Our analysis is different and is able to calculate the flow for any circular obstacle. The condition (3.33) for the initiation of a Taylor column in a stratified fluid is obtained, and from it we show that according to the inviscid quasi-geostrophic theory of stratified flow, each obstacle with a somewhere vertical face induces a Taylor column no matter how low the obstacle is. This is because in rotating stratified flow buoyancy forces inhibit even small, purely vertical motions, and hence fluid particles will tend to flow around, rather than over, vertical faces. Thus the flow induced by a flat-topped cylinder is atypical.

2. Homogeneous Taylor columns

In this section we investigate the flow in a homogeneous system, before investigating the flow in a stratified system and comparing the results obtained in the two different situations. A purely inviscid theoretical flow model is unsatisfactory because, viewed as an initial-value problem, the flow does not settle down to a steady state (Stewartson 1952; Bretherton 1967). There is also a non-uniqueness associated with the closed-streamline regions predicted by inviscid theory. Jacobs (1964) obviated this difficulty by considering the inertialess viscous flow over a circularly symmetric obstacle of fixed height as the perturbation parameter

$$\delta \equiv \nu/fL^2 \tag{2.1}$$

tends to zero. Here ν is the coefficient of kinematic viscosity, f is twice the rotation rate and L a representative horizontal dimension of the obstacle. Jacobs' solution indicates, in agreement with Taylor's experiment, that there exists a stagnant column circumscribing the obstacle. However, as indicated by Jacobs himself, the neglect of the inertia forces is valid only if

$$R \ll \delta^{\frac{1}{2}}, \tag{2.2}$$

where the Rossby number

$$R = V/(fL) \tag{2.3}$$

is based on the oncoming far-upstream velocity V. Condition (2.2) is a severe restriction and does not hold in most geophysical applications. Further, Jacobs' theory indicates that the two-dimensional flow pattern is symmetric to the left and right of a circularly symmetric obstacle looking downstream. Yet this is seen not to be so in most laboratory experiments (Hide & Ibbetson 1966, 1968). Thus, the inclusion of the inertial accelerations would appear essential.

The simplest appropriate model was first formulated by Ingersoll (1969), who considers the quasi-geostrophic inertial flow in the limit of vanishingly small viscosity. The flow in the regions where there are no closed streamlines is then described by the inviscid quasi-geostrophic potential-vorticity equation

$$\frac{D}{Dt}\left(\zeta+\frac{fM}{H}\right) = 0, \tag{2.4}$$

where the relative vorticity

$$\zeta = v_x - u_y = f^{-1}\nabla_h^2 P \tag{2.5 a, b}$$

and $M(\mathbf{x})$, a function of the horizontal vector $\mathbf{x}$, represents the obstacle attached to the lower plane. Equation (2.5*b*) follows from (2.5*a*) by using the geostrophic relationships for the velocities (u, v), and the quantity P, which acts as a stream function for the subsequent two-dimensional motion, represents the total pressure minus the hydrostatic pressure, all divided by the constant density ρ_0. Equation (2.4) expresses the fact that in a rotating, inviscid homogeneous fluid vorticity is generated by vortex stretching (or compression) as represented by the second term in (2.4).

We can most conveniently determine the flow field by non-dimensionalizing all lengths with respect to L, introducing the non-dimensional pressure ψ by

$$P = fLV\psi \tag{2.6}$$

and writing

$$M(\mathbf{x}) = Hh_0 h(\mathbf{r}), \tag{2.7}$$

where $h(\mathbf{r})$ is a non-dimensional function of maximum value unity, so that (2.4) becomes

$$D[\nabla_h^2\psi + h_0 R^{-1}h(\mathbf{r})]/Dt = 0. \tag{2.8}$$

Assuming, for the moment, that all streamlines of the motion originate upstream, we can integrate the nonlinear equation (2.8) between an arbitrary point $\mathbf{r}$ and a point far upstream to obtain

$$\nabla_h^2\psi + h_0 R^{-1}h(\mathbf{r}) = \nabla_h^2\psi_{-\infty}, \tag{2.9}$$

where the subscript $-\infty$ means that the quantity to which it is appended is to be evaluated far upstream. For our assumed uniform flow,

$$\psi_{-\infty} = -y \tag{2.10}$$

and hence (2.9) becomes

$$\nabla_h^2\psi + h_0 R^{-1}h(\mathbf{r}) = 0. \tag{2.11}$$

To (2.11) must be added the upstream boundary condition

$$\nabla\psi \sim (0, -1) \quad (|\mathbf{r}| \to \infty). \tag{2.12}$$

We now assume, for simplicity, that the obstacle is circularly symmetric and use cylindrical polar co-ordinates to express the solution of (2.11) and (2.12) as

$$\psi = -r\sin\theta + \phi(r), \tag{2.13}$$

where

$$r^{-1}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + h_0 R^{-1}h(r) = 0. \tag{2.14}$$

Equation (2.14) can be integrated once, to yield

$$\frac{d\phi}{dr} = -(h_0 R^{-1}/r)\int_0^r th(t)\,dt, \tag{2.15}$$

where the lower limit of the integral in (2.15) must be zero so that $d\phi/dr$ is not infinite at $r = 0$. We note here that $h(r)$ appears in integrated form and so any kinks or sharp corners in h will be 'smoothed' in their influence on ϕ and hence ψ.

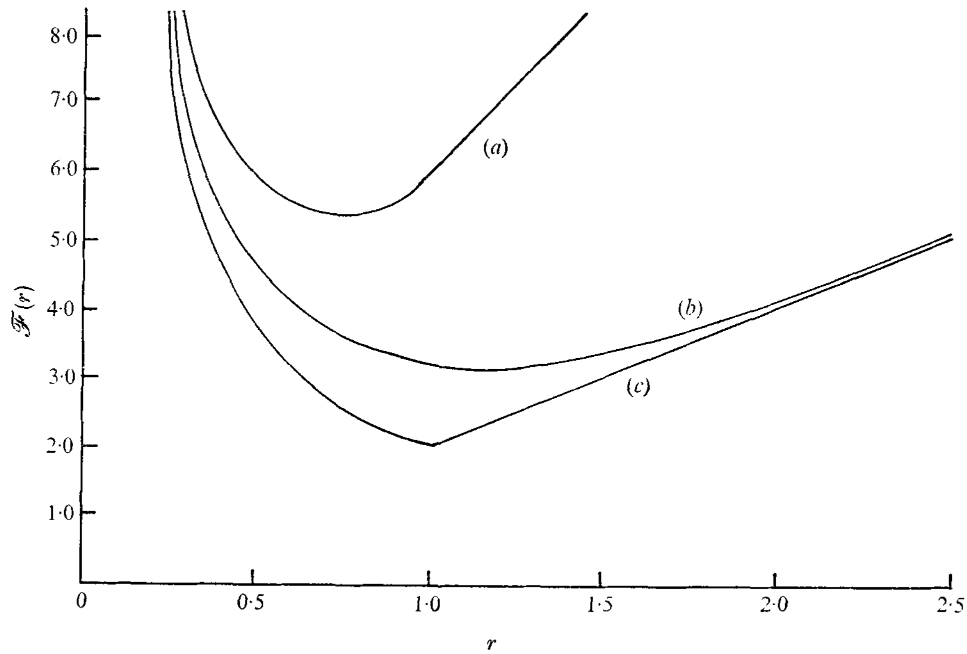

FIGURE 1. The function

$$\mathscr{F}(r) = r\Big/\int_0^r xh(x)\,dx.$$

(a) $h(r) = (1-r)H(1-r)$. (b) $h(r) = e^{-r^2}$. (c) $h(r) = H(1-r)$.

Equation (2.15) can now be integrated once more, ψ determined explicitly and the streamlines determined. However, since we anticipate possible stagnant regions in the flow, it is expedient first to determine if any stagnation points exist. At such points

$$\psi_\theta = \psi_r = 0. \qquad (2.16\,a, b)$$

From (2.13) we see that (2.16 a) can be satisfied only at $\theta = \pm\tfrac{1}{2}\pi$, while differentiating (2.13) and using (2.15), we see that (2.16 b) can be satisfied only at $\theta = -\tfrac{1}{2}\pi$ and at the value(s) of r such that

$$h_0 R^{-1} = r\Big/\int_0^r th(t)\,dt \equiv \mathscr{F}(r), \quad \text{say}, \qquad (2.17), (2.18)$$

$$\sim \begin{cases} [2/h(0)]r^{-1} & (r\rightarrow 0), \qquad (2.19\,a) \\ 2\pi\mathscr{V}^{-1}r & (r\rightarrow\infty), \qquad (2.19\,b) \end{cases}$$

where

$$\mathscr{V} = 2\pi\int_0^\infty th(t)\,dt \qquad (2.20)$$

is the total volume of the obstacle, which will be assumed finite.† Plots of $\mathscr{F}(r)$ for three different obstacles are presented in figure 1.

† For a depression, given by (2.7) with $h_0 < 0$, the correct solution is $\theta = +\tfrac{1}{2}\pi$ and the left-hand side of (2.17) is altered to $-h_0 R^{-1}$.

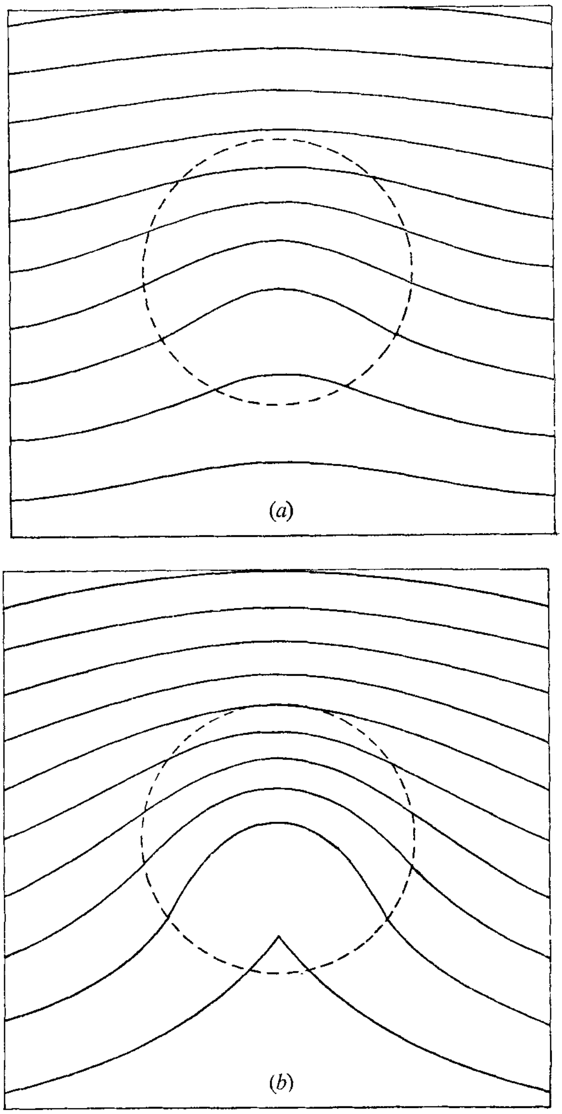

FIGURE 2. The flow of homogeneous fluid over a cone for (a) $h_0R^{-1} = \frac{8}{3}$ and (b) $h_0R^{-1} = \frac{16}{3}$. The stream function is contoured at intervals of 0·4 and the dashed circle indicates the outer edge of the cone at $r = 1$.

If
$$h_0R^{-1} < \min_r \mathscr{F}(r) \equiv \mathscr{F}_{\min}, \quad \text{say}, \tag{2.21 a, b}$$

no stagnation points exist and the flow is given by (2.13) with $\phi(r)$, evaluated from (2.15), given by

$$\phi(r) = -h_0R^{-1}\int_0^r t^{-1}\,dt\int_0^t sh(s)\,ds, \tag{2.22}$$

where the lower limit in the t-integral is arbitrary; changing it from zero to any other finite value only introduces an (arbitrary) constant into the stream

Obstacle	$h(r)$	$B = 0$: $h_0 R^{-1}$	$B = 0$: r_c	$B = \infty$: α	$B = \infty$: r_c
Cylindrical lens	$\begin{cases}(1-r^2) & (r \leqslant 1)\\ 0 & (r \geqslant 1)\end{cases}$	3·67	0·82	1·75	0·79
Hemisphere	$\begin{cases}(1-r^2)^{\frac{1}{2}} & (r \leqslant 1)\\ 0 & (r \geqslant 1)\end{cases}$	2·94	0·93	1·27	1·00
Cylindrical Gaussian	e^{-r^2} $(r < \infty)$	3·13	1·12	2·24	0·86
Flat-topped cylinder	$\begin{cases}1 & (r < 1)\\ 0 & (r > 1)\end{cases}$	2·00	1·00	0	1·00

TABLE 1. The critical height and critical radius for homogeneous flow between horizontal planes ($B = 0$) and for stratified flow in a half-space ($B = \infty$).

function ψ. Such a flow for the particular case of a right circular cone $h(r) = (1-r)H(1-r)$ is depicted in figure 2 (*a*) for $h_0 R^{-1} = \frac{8}{3} = \frac{1}{2}\mathscr{F}_{\min}$.

If
$$h_0 R^{-1} = \mathscr{F}_{\min} \tag{2.23}$$
one stagnation point exists in the flow, and (2.23) represents the highest obstacle, for a given shape $h(r)$, which can exist without a closed-streamline region occurring in the flow. The value of r, r_c say, at which $\mathscr{F}$ attains its minimum, $\mathscr{F}_{\min}$ say, varies quite significantly for different shapes. For example, for a flat-topped cylinder $h(r) = H(1-r)$, $r_c = 1$ and $\mathscr{F}_{\min} = 2$, for a right circular cone, $r_c = 3{\cdot}4$ and $\mathscr{F}_{\min} = 5{\cdot}33$ and for a circular exponential obstacle $h(r) = e^{-r}$, $r_c = 1{\cdot}79$ and $\mathscr{F}_{\min} = 3{\cdot}35$.† Critical values for some other obstacles are given in table 1. The streamline pattern for a right circular cone with $\mathscr{F} = \mathscr{F}_{\min}$ is shown in figure 2 (*b*).

For all obstacles with h_0 less than the critical value, say h_c, the minimum particle speed equals $V[1-(h_0/h_c)]$ and occurs at $r = r_c$, $\theta = -\frac{1}{2}\pi$. The maximum speed, on the other hand, equals $V[1+(h_0/h_c)]$ and occurs at $r = r_c$, $\theta = +\frac{1}{2}\pi$.

Two essential remarks must now be made. First, $\mathscr{F}_{\min}$, and *a fortiori* ψ, depends on integrals of $h(r)$. Hence, obstacles with corners, or other discontinuities, will not lead to atypical results and one can justify the investigation of the flow due to a flat-topped cylinder, for example. Comparing the streamlines in figures 2 (*a*) and (*b*) with those in figures 1 and 2 of Ingersoll (1969) for a flat-topped cylinder clearly illustrates this. Second, $\mathscr{F}_{\min}$ must be positive; it cannot be zero. Thus all shapes must be of finite height before they cause stagnation. While this may appear obvious, it will be shown to be incorrect if the flow is stratified. We also point out that r_c and $\mathscr{F}_{\min}$ are functions only of $h(r)$ for $0 < r < r_c$. Thus the shape of the obstacle can be changed quite drastically for $r > r_c$, as long as $\mathscr{F}(r) > \mathscr{F}_{\min}$ for $r > r_c$, without changing the values of r_c or $\mathscr{F}_{\min}$. In particular, the obstacle can be replaced by one 'chopped' at $r = r_c$ and $h(r)$ set equal to zero for $r > r_c$.

If
$$h_0 R^{-1} > \mathscr{F}_{\min} \tag{2.24}$$

† It is interesting to compare these values with the relationship $h_0 R^{-1} = 2$ deduced by Huppert & Stern (1974) to express the condition when a closed-streamline region first occurs in the homogeneous flow in a rapidly rotating channel with vertical side walls of separation L. The flow takes place between planes which are perpendicular to the rotation and whose separation far upstream is H and far downstream $H(1-h_0)$.

the above analysis indicates the existence of at least two stagnation points and an accompanying closed-streamline region. The closed-streamline portion, however, violates the assumption that all streamlines originate upstream, and the flow field determined by (2.13) and (2.22) is based on a false premise. The problem of determining the flow in a closed-streamline region, to which an upstream boundary condition cannot be applied, was first considered by Batchelor (1956) for a non-rotating system. By determining integral constraints on the equations of motion in the limit of vanishing viscosity, he proves that a steady, inviscid closed-streamline region is one of constant vorticity. Using similar ideas, Ingersoll, conjecturing that there are no high vorticity regions except for the Ekman boundary layers, shows that the closed-streamline region is stagnant and determined by the condition

$$\psi = \text{constant}, \quad \nabla\psi = 0 \tag{2.25}$$

on the boundary of the closed-streamline region. In contrast to the non-rotating case, the closed-streamline region is stagnant because of the existence of Ekman boundary layers and the induced Ekman suction, which can bring the closed-streamline region to rest.

The difficult, free boundary-value problem (2.11), (2.12) and (2.25) has been solved only for a flat-topped cylinder and a relatively high ($h_0 \gg R$), steep-sided cylinder (Ingersoll 1969). The latter solution indicates, as one would expect, that the stagnant region almost completely circumscribes the cylinder. (It would completely circumscribe the cylinder were the walls vertical at the base.) The former solution, for which the stagnant region is a circle of radius $1-2Rh_0^{-1}$ centred on $(0, -2Rh_0^{-1})$, is, however, somewhat special. This is because the vortex lines of the flow, once they have been compressed by the amount h_0, suffer no further compression as they move over the obstacle. This will not be true for most other obstacles for which the critical stagnation point occurs at a point below the obstacle's maximum height. Thus further compression is needed for some of the vortex lines to be able to flow over the obstacle. The extent and shape of the Taylor column hence differs significantly for different obstacles. Thus, if the present ideas of Taylor columns are to be used in explaining geophysical situations it is essential to investigate this problem further and it is planned to do so in a subsequent paper.

3. Stratified Taylor columns

Almost all geophysically interesting flows are heterogeneous and it is hence of interest to investigate the effects of stratification. We do so in this section by an analysis which parallels that used in the last section. One of the conclusions, however, is fundamentally different. We find, in contrast to the homogeneous situation, that an obstacle with a vertical face induces a closed-streamline region no matter how low the obstacle.

We consider exactly the same flow geometry as before except for the addition of a basic stratification which far upstream has constant Brunt–Väisälä frequency

$$N = (g\beta)^{\frac{1}{2}}. \tag{3.1}$$

Because a stratified flow can support vertical variations of the horizontal velocity, a vertical shear in the upstream velocity gradient could be easily incorporated. However, this effect adds very little, either qualitatively or quantitatively (Hogg 1973), unless the basic velocity profile is zero at some height within the channel. Then there is the possibility of a stationary baroclinic standing wave, an effect partially investigated by de Szoeke (1972).

We momentarily confine attention to the flow in a channel rather than in a half-space, which would be the more relevant model for strong stratification, $N \gg fL/H$, and show below how the solutions for a channel include those for a half-space.

Writing the total density and pressure as

$$\tilde{\rho} = \rho_0(e^{-\beta z} + \rho) \tag{3.2}$$

and

$$\tilde{P} = p/\rho_0 = (g/\beta)\, e^{-\beta z} + P, \tag{3.3}$$

we write the quasi-geostrophic potential-vorticity equation (Phillips 1963) as

$$\frac{D}{Dt}\left[(f+\zeta)\frac{\partial \tilde{\rho}}{\partial z}\right] = 0, \tag{3.4}$$

the vertical momentum equation as

$$P_z = -g\rho, \tag{3.5}$$

and the density conservation equation as

$$\mathbf{q}\,.\,\nabla\rho - \beta w = 0, \tag{3.6}$$

where we have made the Boussinesq approximation and assumed that the flow is non-diffusive and in hydrostatic balance $[V^2/(N^2L^2) \ll 1]$.

Assuming tentatively that all streamlines originate upstream, we can integrate (3.4) to obtain

$$\zeta - (f/\beta)\frac{\partial \rho}{\partial z} = 0, \tag{3.7}$$

indicating that quasi-geostrophic vorticity in a stratified fluid is generated by isopycnal separation, which in its turn is caused by the presence of the obstacle. Substituting the geostrophic relationship between ζ and P into (3.7) and relating the density gradient to the pressure by (3.5), we obtain the governing equation

$$\nabla_h^2 P + (f^2/N^2)\, P_{zz} = 0. \tag{3.8}$$

The lower boundary condition

$$w = \mathbf{q}\,.\,\nabla M \quad \text{(on the obstacle)} \tag{3.9}$$

can be written in terms of P by using (3.5) and (3.6) to obtain

$$P_z = -N^2 M \quad \text{(on the obstacle).} \tag{3.10}$$

Anticipating that we shall obtain a closed-streamline region for comparatively small obstacles $[h_0/H = O(R)]$, we linearize (3.10) and write

$$P_z = -N^2 M \quad (z = 0). \tag{3.11}$$

The upper boundary condition, without linearization, becomes

$$P_z = 0 \quad (z = H). \tag{3.12}$$

Non-dimensionalizing horizontal lengths with respect to L, introducing a non-dimensional vertical co-ordinate η by

$$z = H\eta, \tag{3.13}$$

and using the same definitions of ψ and h as in §2, we see that the stratified flow field can be determined from the boundary-value problem

$$\psi_{rr} + r^{-1}\psi_r + B^{-2}\psi_{\eta\eta} = 0, \tag{3.14}$$

$$\psi_\eta = \begin{cases} -h_0 B^2 R^{-1} h(r) & (\eta = 0), \quad (3.15) \\ 0 & (\eta = 1), \quad (3.16) \end{cases}$$

$$\psi \sim -r\sin\theta \quad (r \to \infty,\ \pi \geqslant |\theta| > \tfrac{1}{2}\pi), \tag{3.17}$$

where

$$B \equiv NH/fL. \tag{3.18}$$

Hogg (1973) solved these equations for the special case $h(r) = H(1-r)$ by expressing the solution as an infinite sum of trigonometric eigenfunctions in the vertical and modified Bessel functions in the horizontal. There are some difficulties with this form of solution, including the fact that the representation has a Gibbs phenomenon just at the point $z = 0$, where the boundary condition (3.15) is applied. We find it more suitable to represent the solution in terms of a Hankel transform in r, which allows us to determine the solution for arbitrary $h(r)$. By this means the solution of (3.14)–(3.17) can be written as

$$\psi(r,\theta,\eta) = -r\sin\theta + h_0 B R^{-1} \int_0^\infty g(t,\eta)\,\hat{h}(t)\,J_0(tr)\,dt, \tag{3.19}$$

where the kernel

$$g(t,\eta) = \cosh Bt(1-\eta)/\sinh Bt \tag{3.20}$$

and $\hat{h}(t)$ is the zeroth-order Hankel transform of $h(r)$, given by

$$\hat{h}(t) = \int_0^\infty r h(r)\,J_0(tr)\,dr. \tag{3.21}$$

We note that $g(t,\eta)$ decreases exponentially with increasing η, indicating that the disturbance caused by the obstacle decreases rapidly with increasing height. While this is also true of Hogg's solution, it is obscured by the fact that his solution is represented by an (infinite) sum of terms varying sinusoidally in the vertical.

Before proceeding, we briefly consider the two limits $B \to 0$, which corresponds to homogeneous flow, and $B \to \infty$, which corresponds to strong stratification. In the former case

$$g(t,\eta) \sim (Bt)^{-1} \quad (B \to 0), \tag{3.22}$$

which, when substituted into the partial differential of (3.19) with respect to r, indicates upon using (3.21) that

$$\psi_r \sim -\sin\theta - h_0 R^{-1} \int_0^\infty \int_0^\infty x h(x)\,J_0(tx)\,J_1(tr)\,dt\,dx. \tag{3.23}$$

Evaluating the integral with respect to t in (3.23) using relationship 11.303 for the integral of two Bessel functions given in Wheelon (1968) we obtain

$$\psi_r \sim -\sin\theta - (h_0 R^{-1}/r)\int_0^r xh(x)\,dx \quad (B \to 0). \tag{3.24}$$

Integrating (3.24) with respect to r, we see that the result is in agreement with the relationships (2.16) and (2.19) obtained in the previous section.

Remembering that a quasi-geostrophic analysis is formally valid for $R \ll 1$, we can appreciate the sense of the $B \to 0$ solution by stating that the flow is effectively homogeneous, and the results of §2 apply, if both B and R are very much less than unity. If B is not very much less than unity the flow is heterogeneous and the results of this section apply. A rigorous justification of these statements can be obtained in a straightforward manner by considering an ordered expansion of the equations of motion as a power series in R. If B is $o(1)$ with respect to R the vertical gradients of the zeroth-order pressure and horizontal velocities are zero, and the zeroth-order flow is independent of B. If B is $O(1)$ with respect to R the zeroth-order gradients are non-zero, and the flow depends on B in the manner described in this section.

We turn now to the strong stratification limit, for which

$$g(t,\eta) \sim e^{-B\eta t} \quad (B \to \infty) \tag{3.25}$$

and

$$\psi \sim -r\sin\theta + h_0 B R^{-1}\int_0^\infty \hat{h}(t)\, e^{-B\eta t} J_0(tr)\,dt \quad (B \to \infty). \tag{3.26}$$

This solution can be used to describe the stratified flow in a half-space by interpreting the limit $B \to \infty$ to mean $H \to \infty$ with N/fL fixed. Equation (3.26) then becomes

$$\psi = -\ r\sin\theta + (Nh_*/V)\int_0^\infty \hat{\hbar}(t)\exp[-(Nz/fL)t]\, J_0(tr)\,dt, \tag{3.27}$$

valid in the half-space $0 \leqslant z < \infty$, where the obstacle shape normalization (2.4) has been replaced by

$$M(\mathbf{x}) = h_*\hbar(r), \tag{3.28}$$

with unity as the maximum value of the dimensionless shape function $\hbar(r)$, and $\hat{\hbar}(t)$ is the Hankel transform of $\hbar(r)$ in the style of (3.21).

Returning to the stratified flow between two horizontal planes, we investigate the existence of possible stagnation points in the flow by determining the points of zero horizontal velocity, and in consequence of (3.6), zero vertical velocity. We do this by evaluating the zeros of the derivatives of (3.19) with respect to both r and θ. These are zero at $\theta = -\frac{1}{2}\pi$ and $\eta = \eta_0$ provided that

$$h_0 R^{-1} > \min_r \left(B\int_0^\infty t g(t,\eta_0)\,\hat{h}(t)\, J_1(tr)\,dt\right)^{-1} \tag{3.29}$$

$$\equiv \min_r \mathscr{G}(r,\eta_0), \quad \text{say}. \tag{3.30}$$

Since $\mathscr{G}(r, \eta_0)$ is an increasing function of η_0, a stagnation point occurs first at the bottom, $\eta_0 = 0$, at that value of r which minimizes $\mathscr{G}(r, 0)$; that is when

$$h_0 R^{-1} = \min_r \mathscr{G}(r, 0) \equiv \mathscr{G}_{\min}, \quad \text{say,} \tag{3.31), (3.32}$$

$$\sim \begin{cases} \mathscr{F}_{\min} & (B \to 0), \\ \alpha / B & (B \to \infty), \end{cases} \tag{3.33), (3.34}$$

where

$$\alpha = \min_r \left(\int_0^\infty t \hat{h}(t) \, J_1(tr) \, dt \right)^{-1}. \tag{3.35}$$

Condition (3.32) is the extension of the stagnation condition (2.23) when the fluid is stratified. The important difference between the two conditions is that, while $\mathscr{F}_{\min}$ is always positive, $\mathscr{G}_{\min}$ can be zero. This occurs whenever the obstacle has a vertical face and such obstacles, no matter how low, will always lead to a closed streamline region in the flow.

Mathematically, the closed streamline occurs because the discontinuity at $r = r_0$, say, in $h(r)$ associated with the vertical face makes the integral in (3.29) divergent at $r = r_0$, $\eta = 0$, and hence $\mathscr{G}_{\min} = 0$. This is proved in the appendix. Physically, the closed streamline occurs because in a rotating stratified fluid vortex lines cannot be stretched, or compressed, over an infinitesimal horizontal distance without the vertical velocity becoming infinite. Alternatively, the effects of rotation and stratification constrain the influence of variations on a horizontal length scale l to a vertical scale of order fl/N, which becomes arbitrarily small as $l \to 0$. Thus, the fluid near the ground flows around the vertical face. This can be seen from the equations by appreciating that a discontinuity in h, and thus in M, in (3.11) implies a discontinuity in P_z at $z = 0$. From (3.5) this implies a discontinuity in ρ, which by (3.6) implies that, if w is to be finite at r_0, the radial velocity must be zero there. Thus there is a closed streamline at $r = r_0$ and $z = 0$. While the existence of a closed streamline negates our initial assumption that all streamlines originate upstream, the above proves rigorously that no flow without closed streamlines exists when $h(r)$ has a discontinuity.

Another situation in which topographic discontinuities exert a dominant influence on the flow has been discussed by Miles & Huppert (1969), who consider two-dimensional non-rotating flow of a stratified fluid over an obstacle. They show that in this non-rotating case every obstacle with a vertical face leads to overturning and closed streamlines somewhere in the flow and conclude that models of lee waves generated by vertical-faced obstacles include features which are not present for smooth obstacles of sufficiently small height.

We have shown how to calculate $\mathscr{G}_{\min}$ and note that flows for $h_0 R^{-1} \leqslant \mathscr{G}_{\min}$ are similar to homogeneous flows for $h_0 R^{-1} \leqslant \mathscr{F}_{\min}$. The only essential differences are that in a stratified flow there is a small vertical velocity and that the deviation of the flow from a uniform flow decreases with height. The streamlines in a horizontal plane for the stratified flow over a cone with $h_0 R^{-1} = \mathscr{G}_{\min}$ are at $z = 0$ similar to those in figure 2 (*b*) and at larger values of z similar to those in figure 2 (*a*). For flow over any obstacle with $h_0 R^{-1} \leqslant \mathscr{G}_{\min}$, the minimum velocity occurs at $(r, \theta, \eta) = (r_c, -\frac{1}{2}\pi, 0)$ and is equal to $V[1 - (h_0 R^{-1}/\mathscr{G}_{\min})]$, while the maximum

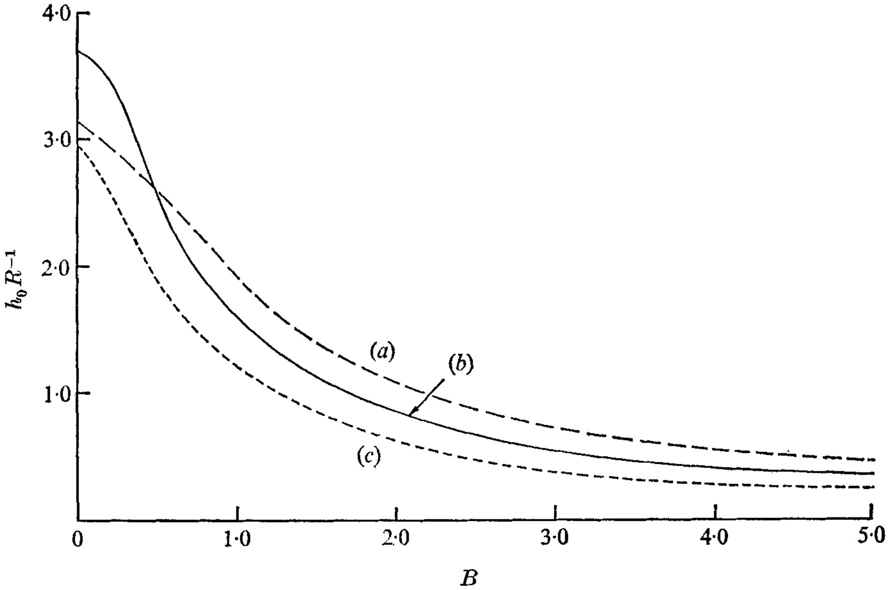

FIGURE 3. The non-dimensional critical height h_0R^{-1} versus B. (a) $h(r) = e^{-r^2}$. (b) $h(r) = (1-r^2)H(1-r)$. (c) $h(r) = (1-r^2)^{\frac{1}{2}}H(1-r)$.

velocity occurs at $(r_c, +\frac{1}{2}\pi, 0)$ and is equal to $V[1+h_0R^{-1}/\mathscr{G}_{\min}]$. Hence at critical conditions the maximum flow speed is twice the free-stream speed.

It is now natural to ask: what is the form of the region of closed stream surfaces for $h_0R^{-1} > \mathscr{G}_{\min}$? The proper resolution of the three-dimensional flow under this condition is not yet known. A simple extension of the ideas used in the homogeneous two-dimensional problem as described initially by Ingersoll does not appear possible and I am unable at present to determine what replaces his conditions when the flow is stratified. A possible line of advance might be to extend the solution (3.19) beyond the limits upon which it is based and conjecture that, provided that the time since initiation of the flow is short compared with the stratified spin-down time $L(V\Omega)^{-\frac{1}{2}}$, equation (3.19) will predict the form of flow fairly well. This line of argument seems highly questionable because, using the homogeneous flow over a flat-topped cylinder as a special test case, we indicated in the previous section that in the inviscid limit the closed-streamline region consists of a circle, centre $(0, -2Rh_0^{-1})$ and radius $1-2Rh_0^{-1}$, which for large values of h_0R^{-1} approaches the circle circumscribing the cylinder. However, the purely inviscid solution (2.13) and (2.21) implies that the closed-streamline region is very much larger; in fact for $h_0R^{-1} > 3{\cdot}59$ the closed-streamline region encompasses the whole cylinder as well as a considerably larger area. The purely inviscid solution hence differs considerably from the solution determined from the analysis in the limit of vanishing viscosity. The resolution of this situation is at present being investigated by considering the initial-value problem and we anticipate discussing the results elsewhere in the near future.

Since the flow in a half-space can be obtained by evaluating the strong stratification limit for flow between horizontal planes, it follows that the characteristics determined above for the flow between horizontal planes will also apply to the flow in a half-space. In particular, a closed streamline will occur for an infinitesimally small obstacle if it has a vertical face. If the obstacle does not have a vertical face, the critical value of Nh_*/V is given by the value of α defined by (3.34) and tabulated for some specific examples in table 1.

Curves of the critical value of h_0R^{-1} as a function of B for three particular obstacles are presented in figure 3 and the critical conditions at zero B and infinite B tabulated in table 1. For very small values of B the flow is almost homogeneous and the critical height for an obstacle is roughly proportional to the reciprocal of its non-dimensional volume $\mathscr{V}$. As B increases the disturbance is increasingly confined to the bottom and the critical height decreases. For $B \gg 1$ the penetration depth fL/N is much less than the actual depth H, and the critical height varies inversely with B. This is a consequence either of (3.34), or more directly, from the dimensional argument that, for large B, Hh_0 must be functionally related to B in such a way as to be independent of the value of H.

4. Conclusions

One of the aims of these remarks has been to indicate that by developing an understanding of the flow of homogeneous fluid over an isolated obstacle in a rotating system one can proceed quite some way in understanding and explaining the effects present if the flow is stratified. There is the fundamental and important difference, however, that obstacles with vertical faces, which might be used to model realistic obstacles, do not induce flows qualitatively similar to those induced by smooth obstacles in this stratified situation. Vertical-faced obstacles, no matter how low, induce a closed-streamline region and an associated Taylor column, which are not necessarily present in the flow over a smooth obstacle of comparable height.

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Appendix

We prove here that any cylindrical obstacle with a vertical face has a critical value of h_0R^{-1} equal to zero. Mathematically, we need to prove that a discontinuity in h implies that $\mathscr{G}_{\min} = 0$; that is, the integral involved in the definition of $\mathscr{G}(r, 0)$,

$$\int_0^\infty t \coth(Bt)\,\hat{h}(t)\,J_1(tr)\,dt \equiv I(r), \quad \text{say,} \tag{A 1}$$

is divergent for at least one value of r.

Denoting the integrand of $I(r)$ as $f(t)$, so that

$$f(t) = t \coth(Bt)\,\hat{h}(t)\,J_1(tr), \tag{A 2}$$

we see that, since $f(t)$ is bounded in $(0, \infty)$, $I(r)$ can only be infinite if $f(t)$ is not sufficiently well behaved at either $t = 0$ or $t = \infty$.

Considering the former case first, we note that

$$f(t) \sim B^{-1}\hat{h}(0)\,tr \quad (t \to 0), \tag{A 3}$$

where, from (3.21),

$$\hat{h}(0) = \int_0^\infty xh(x)\,dx = \mathscr{V}/2\pi. \tag{A 4}$$

Thus $f(t)$ is integrable at the origin.

Turning now to the limit of infinite t, we note that

$$f(t) = t\hat{h}(t)\,J_1(tr)\,[1 + O(e^{-2Bt})] \quad (t \to \infty). \tag{A 5}$$

We now assume that $h(r)$ has at most one discontinuity, at $r = r_0$ say, such that

$$h(r) = \begin{cases} h_-(r_0) + (r - r_0)\,h'_-(r_0) + \dots & (r \uparrow r_0), \quad \text{(A 6)} \\ h_+(r_0) + (r - r_0)\,h'_+(r_0) + \dots & (r \downarrow r_0), \quad \text{(A 7)} \end{cases}$$

and express $\hat{h}(t)$ as

$$\hat{h}(t) = \left(\int_0^{r_0} + \int_{r_0}^\infty\right)[xh(x)\,J_0(tx)]\,dx. \tag{A 8}$$

Integrating (A 8) twice by parts using standard Bessel-function relationships and the expressions (A 6) and (A 7), we can write

$$\begin{aligned} \hat{h}(t) = {} & t^{-1}r_0[h_-(r_0) - h_+(r_0)]\,J_1(tr_0) \\ & - t^{-2}r_0[h'_-(r_0) - h'_+(r_0)]\,J_2(tr_0) \\ & + t^{-2}\left(\int_0^{r_0} + \int_{r_0}^\infty\right)\{[xh''(x) - h'(x)]\,J_2(tx)\}\,dx. \end{aligned} \tag{A 9}$$

Substituting (A 9) into (A 5), we see that

$$\begin{aligned} f(t) \sim {} & r_0[h_-(r_0) - h_+(r_0)]\,J_1(tr_0)\,J_1(tr) \\ & + t^{-1}r_0[h'_-(r_0) - h'_+(r_0)]\,J_2(tr_0)\,J_1(tr) \quad (t \to \infty). \end{aligned} \tag{A 10}$$

Recalling that

$$\int^\infty J_1(tr_0)\,J_1(tr)\,dt$$

diverges at $r = r_0$, while

$$\int^\infty t^{-1}J_2(tr_0)\,J_1(tr)\,dt$$

is finite for all r we see from (A 10) that $I(r)$ is infinite at $r = r_0$ if $h_-(r_0) \neq h_+(r_0)$; that is unless h is continuous at r_0. We also see that discontinuities in the slope of $h(r)$ lead to finite values of $I(r)$.

If $h(r)$ has more than one discontinuity, at $r = r_{01}$, $r = r_{02}$, ..., say, the above argument can be easily extended to prove that $I(r)$ is infinite at $r = r_{01}$, $r = r_{02}$, ...

We note finally that the ability to expand $h(r)$ about r_0 in a regular Taylor series of the form (A 6) and (A 7) is not necessary to be able to conclude that obstacles with a vertical face, and only those with a vertical face, have a zero critical value of $h_0 R^{-1}$. The result can be shown to be correct if

$$h(r) = \begin{cases} h_-(r_0) + O(r-r_0)^{\mu_-} & (r \uparrow r_0), \quad \text{(A 11)} \\ h_+(r_0) + O(r-r_0)^{\mu_+} & (r \downarrow r_0), \quad \text{(A 12)} \end{cases}$$

for positive μ_- and μ_+.

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